

hep-th/0108122

INSTANTONS, THE INFORMATION METRIC, AND THE AdS/CFT CORRESPONDENCE

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Abstract

We describe some remarkable properties of the so-called Information Metric on instanton moduli space. This Metric is manifestly gauge and conformally invariant and coincides with the Euclidean AdS_5 -metric on the one-instanton $SU(2)$ moduli space for the standard metric on \mathbb{R}^4 . We propose that for an arbitrary boundary metric the AdS/CFT bulk space-time *is* the instanton moduli space equipped with the Information Metric.

To test this proposal, we examine the variation of the instanton moduli and the Information Metric for first-order perturbations of the boundary metric and obtain three non-trivial and somewhat surprising results: (1) The perturbed Information Metric is Einstein. (2) The perturbed instanton density is the corresponding massless boundary-to-bulk scalar propagator. (3) The regularized boundary-to-bulk geodesic distance is proportional to the logarithm of the perturbed instanton density. The Hamilton-Jacobi equation implied by (3) equips the moduli space with a rich geometrical structure which we explore.

These results tentatively suggest a picture in which the one-instanton sector of $SU(2)$ Yang-Mills theory (rather than some large- N limit) is in some sense holographically dual to bulk gravity.

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1 INTRODUCTION

A large amount of work has been done on the AdS/CFT correspondence [1, 2, 3, 4], in particular in the context of supergravity on $AdS_5 \times S^5$ and its dual relationship to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theories on the boundary of AdS_5 in the large N limit. It was realized early on that instantons of the boundary gauge theory play a central role in this correspondence [5].

In particular, in a very interesting series of papers, Dorey et al [6] were able to compute non-perturbative corrections due to Yang-Mills instantons and show that in the bulk $AdS_5 \times S^5$ they correspond to the contributions of D(-1)-brane instantons in type IIB theory to the R^4 couplings. What is remarkable is that, in the large N saddle point approximation, the single super-instanton moduli space collapses to $AdS_5 \times S^5$ with AdS_5 arising as the instanton number $k = 1$ $SU(2)$ moduli space. This work was generalized to other situations with less supersymmetry. For example, in $\mathcal{N} = 2$ theories arising in Type I' D3 branes, the Yang-Mills instanton sees the relevant bulk geometry $AdS_5 \times S^5/Z_2$ as well as the geometry of D-7 branes appearing in the Type I' vacuum [7]. Related D-instanton probe calculations appear in [8].

These results suggest that Yang-Mills instantons are a good probe of the bulk geometry in general. More precisely, the instanton moduli space in the large N saddle point limit becomes the bulk geometry. In particular the AdS_5 coordinates are provided by the position and the scale of the instanton, with zero scale (UV regime) being the boundary of AdS_5 and large scale (IR regime) being the deep interior. In obtaining this, one integrates out the zero modes corresponding to the gauge orientation of the instanton in $SU(N)$ (whose number grows linearly with N) leaving effectively just the moduli associated with $SU(2)$ instantons. The details of the internal space (such as S^5) depend on the R-symmetry of the problem and hence the number of supersymmetries. These coordinates appear explicitly as certain bilinears of fermionic zero modes.

In the light of these results it is natural to ask if the feature that instanton moduli space gives rise to the bulk geometry persists in situations where Yang-Mills theory on the D3-branes couples to non-trivial bulk field backgrounds. For example, if the 4-dimensional space-time on which Yang-Mills theory lives is curved, then does the instanton moduli space give rise to the deformed AdS_5 which is Einstein and whose metric approaches the 4-dimensional metric as one goes to the boundary? The AdS/CFT correspondence together with the instanton probe idea would suggest that this is the case. The purpose of this paper is to study this problem.

As mentioned earlier, in the large N saddle point effectively only the moduli associated with $SU(2)$ instantons survive. We will therefore analyze this problem by directly

working with these $SU(2)$ instantons. The first question to settle is what is the metric on the $k = 1$ $SU(2)$ instanton moduli space one should consider. Even in the flat case this is not *a priori* clear as the usual L^2 -metric on the moduli space will clearly not give rise to the AdS_5 metric. Indeed the isometries of AdS_5 are related to conformal invariance on the boundary while the L^2 -metric is not conformally invariant.

Remarkably, there exists a metric on the moduli space, called the Information Metric, first suggested in the moduli space context in [9], which is manifestly gauge and conformally invariant. It is defined by

$$G_{AB}(y) \sim \int \sqrt{g} d^4x F^2(x; y) \partial_A \log F^2(x; y) \partial_B \log F^2(x; y) . \quad (1.1)$$

Here ∂_A is the derivative with respect to the instanton moduli y^A and $F^2(x; y)$ is shorthand for the instanton density. It follows from symmetry arguments [9] or by explicit calculation (section 2.2) that for the flat metric on \mathbb{R}^4 this metric, in fact, gives the $k = 1$ $SU(2)$ instanton moduli space the geometry of AdS_5 .

The purpose of this paper is to propose a central role for the Information Metric on instanton moduli space in the context of the AdS/CFT correspondence. Specifically, we propose that for an arbitrary boundary metric the AdS/CFT bulk space-time *is* the instanton moduli space equipped with the Information Metric.

Our aim is thus to study instanton solutions on a curved 4-dimensional space (which is topologically \mathbb{R}^4 or S^4) and compute the corresponding Information Metric on the moduli space. For a general boundary metric this is a difficult task. So to test the proposal, we examine the variation of the instanton moduli and the instanton density F^2 and the induced variation of the Information metric for first-order perturbations of the boundary space-time metric away from the (conformally) flat metric. We obtain a number of interesting and technically as well as mathematically quite surprising results, namely that to this first order in the metric perturbation

1. the Information metric is Einstein and approaches the perturbed space-time metric on the boundary - in other words, the variation of the Information Metric *is* the boundary-to-bulk graviton propagator;
2. for this perturbed bulk metric, the perturbed instanton density $F^2(x; y)$ is the boundary-to-bulk massless scalar propagator from the point x at the boundary to the point y in the bulk;
3. the regularized geodesic distance between a point x at the boundary and an arbitrary point in the interior labelled by the instanton moduli y is proportional to the logarithm of the perturbed instanton action density $F^2(x; y)$.

We also show that result (2) and (3) above imply (1) to first order, and that to the second order not all the three results above can simultaneously hold. It would be interesting to see what exactly happens at the next order, but the techniques used in this paper are too cumbersome to apply.

The fact that these results are true to first order by itself is quite remarkable. (1) already shows that the Information Metric is the ‘correct’ bulk metric to first order in the metric perturbation. And (2) and (3) show that it is indeed extremely natural to think of the bulk metric as the Information Metric on the instanton moduli space. It somehow suggests that self-dual solutions with instanton number $k = 1$ of $SU(2)$ Yang-Mills theory are holographically dual to bulk gravity theory, with the choice of Information metric being dictated by 4-dimensional conformal invariance which (as in string theory) implies the bulk (target-space) Einstein equations. We will comment on this scenario in the conclusion.

The paper is organized as follows. In section 2, we briefly review the Information metric for the unperturbed case. We will see that for the unperturbed case all the three results stated above are true. We will also obtain some useful identities that will be used later in the paper. In section 3, we discuss the first order correction to the instanton solution and compute the corresponding correction to the Information metric. We will show that all the three results hold up to this order. In section 4, we discuss the general structure of the geometry of moduli space implied by the result (3) or, more precisely, by the Hamilton-Jacobi equation following from it. We will then explore the relations between the above three facts and in particular show that result (1) is implied by (2) and (3). We will also obtain certain harmonic coordinates on the moduli space which are expressed in terms of the action density and its x -derivatives. Our results certainly raise more questions than they answer, so in section 5 we address some of these issues.

2 THE INFORMATION GEOMETRY OF THE UNPERTURBED INSTANTON MODULI SPACE \mathcal{M}_0

2.1 THE INFORMATION METRIC ON THE INSTANTON MODULI SPACE

The most important property of the Information Metric

$$\begin{aligned} G_{AB}(y) &\sim \int \sqrt{g} d^4x F^2(x; y) \partial_A \log F^2(x; y) \partial_B \log F^2(x; y) \\ &= \int \sqrt{g} d^4x \frac{\partial_A F^2(x; y) \partial_B F^2(x; y)}{F^2(x; y)} \end{aligned} \quad (2.1)$$

($F^2 \sim g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}^a F_{\alpha\beta}^a$ - later on we will fix the precise normalizations we are going to use) for our purposes is that it inherits all the space-time symmetries of the instanton density

and equations. In particular, it is completely gauge invariant and hence degenerate along the directions in the moduli space corresponding to global gauge rotations. It also depends only on the conformal structure, i.e. the conformal equivalence class of the space-time metric $g_{\mu\nu}$. This is manifest from the expression (2.1) as both $\sqrt{g}F^2$ and the logarithmic derivatives of F^2 are conformally invariant.

This construction of a metric on the instanton moduli space is a special case of a general construction of a metric on a space of probability densities. This *Fisher Information Metric* has numerous applications in statistics and probability theory - for details and references see e.g. [10, 11].

Clearly (2.1) is rather different from the standard L^2 -metric

$$g_{AB}(y) \sim \int \sqrt{g} d^4x g^{\mu\nu}(x) \partial_A A_\mu^a(x; y) \partial_B A_\nu^a(x; y) \quad (2.2)$$

on the instanton moduli space which is not conformally invariant but is non-degenerate in the direction of global gauge moduli.

The properties described above make the Information Metric an extremely natural object to consider when studying the geometry of moduli spaces. However, these attractive features are somewhat offset by the fact that it appears to be quite difficult in general to establish if or when the Information Metric is non-degenerate (as defined in (2.1) it only gives a semi-positive-definite quadratic form) and this may be the reason why it has received little attention in the differential-geometric context - the only articles we are aware of are [9, 12, 10].

Moreover, even if one can prove in certain cases that the Information Metric is indeed non-degenerate, an explicit or reasonably general closed form expression for the inverse metric is hard to come by. For the same reason, calculations of the curvature of the metric are difficult and practically nothing seems to be known in cases where the Information Metric is not known explicitly. Here we will bypass this problem (and determine the curvature of the Information Metric, among other things) by working with a perturbation of the Information Metric around a known background.

2.2 EXPLICIT EVALUATION OF THE INFORMATION METRIC ON \mathcal{M}_0

Let us now consider specifically the Information Metric on the instanton moduli space $\mathcal{M}_{k=1}(\mathbb{R}^4, SU(2))$ of $k = 1$ instantons on \mathbb{R}^4 with the standard metric. We will denote this space by \mathcal{M}_0 , the subscript indicating that this is the moduli space associated to the standard flat metric on \mathbb{R}^4 or (by conformal invariance) the standard round metric on S^4 .

This space is known to be five-dimensional, with the topology of a ball, thus parametrized by five moduli, namely four coordinates a^μ (the centre of the instanton) and one scale ρ . As the instanton equations are invariant under conformal $SO(5,1)$ transformations of \mathbb{R}^4 , the conformal group acts via isometries on \mathcal{M}_0 equipped with the Information Metric. Thus, the argument goes [9], the Information Metric is the (unique up to a scale) $SO(5,1)$ -invariant metric on the five-ball, i.e. the hyperbolic (Euclidean AdS) metric. Let us check this explicitly.

Explicit expressions for the field strength (in the regular gauge) and the instanton density as functions of the moduli $y^A = (\rho, a^\mu)$ are

$$F_{\mu\nu}^a(x; \rho, a^\mu) = -4\eta_{\mu\nu}^a \frac{\rho^2}{[(x-a)^2 + \rho^2]^2} \quad (2.3)$$

$$\text{tr } F^2(x; \rho, a^\mu) = 96 \frac{\rho^4}{[(x-a)^2 + \rho^2]^4} . \quad (2.4)$$

Here $\eta_{\mu\nu}^a$ are the 't Hooft eta-symbols, a basis for self-dual two-forms on \mathbb{R}^4 - see (3.16).

Let us define

$$F^2 := \frac{\rho^4}{[(x-a)^2 + \rho^2]^4} . \quad (2.5)$$

Since

$$\frac{6}{\pi^2} \int d^4x F^2 = 1 \quad (2.6)$$

we define the Information Metric by

$$G_{AB} = c \frac{6}{\pi^2} \int d^4x F^2 \partial_A \log F^2 \partial_B \log F^2 , \quad (2.7)$$

where c is a constant which we will choose in a convenient way below.

The logarithmic derivatives $\partial_A \log F^2$ of F^2 with respect to the moduli are

$$\begin{aligned} \partial_\rho \log F^2 &= 4 \left[\frac{1}{\rho} - \frac{2\rho}{(x-a)^2 + \rho^2} \right] \\ \partial_{a^\mu} \log F^2 &= 8 \frac{(x-a)_\mu}{(x-a)^2 + \rho^2} . \end{aligned} \quad (2.8)$$

One then finds

$$G_{a^\mu a^\nu} = c \frac{16}{5} \frac{\delta_{\mu\nu}}{\rho^2} , \quad G_{\rho\rho} = c \frac{16}{5} \frac{1}{\rho^2} , \quad G_{a^\mu \rho} = 0 . \quad (2.9)$$

This establishes that the Information Metric on \mathcal{M}_0 is

$$ds_0^2 \equiv G_{AB} dy^A dy^B = \frac{16c}{5} \frac{d\rho^2 + d\vec{a}^2}{\rho^2} . \quad (2.10)$$

Choosing $c = 5/16$, i.e.

$$G_{AB} = \frac{15}{8\pi^2} \int d^4x F^2 \partial_A \log F^2 \partial_B \log F^2 , \quad (2.11)$$

we thus get precisely the unit hyperbolic metric. This is a smooth conformally invariant and geodesically complete metric on \mathcal{M}_0 . We will adopt this normalization for the Information Metric in general.

By contrast, the L^2 -metric on \mathcal{M}_0 for \mathbb{R}^4 , the flat metric

$$ds^2 = d\rho^2 + d\vec{a}^2 \quad (2.12)$$

with a singularity at $\rho = 0$ corresponding to zero-size instantons, is neither conformally invariant nor geodesically complete. Also the L^2 -metric for S^4 is quite different - and somewhat more complicated. It has been studied in [13]. Note that the Information Metric, on the other hand, is of course the same for \mathbb{R}^4 and S^4 because of conformal invariance!

Let us also note that the Information Metric on the moduli space $\mathcal{M}_{k=1}(\mathbb{R}^4, SU(N))$ of $k = 1$ $SU(N)$ -instantons collapses to that on

$$\mathcal{M}_0 = \mathcal{M}_{k=1}(\mathbb{R}^4, SU(2)) \subset \mathcal{M}_{k=1}(\mathbb{R}^4, SU(N)) . \quad (2.13)$$

Indeed, for $k = 1$ and gauge group $SU(N)$ one can construct all instanton solutions by embedding the $SU(2)$ -instanton inside $SU(N)$ and acting with rigid $SU(N)$ gauge transformations. The L^2 -metric on the resulting $4N$ -dimensional moduli space $\mathcal{M}_{k=1}(\mathbb{R}^4, SU(N))$ is non-degenerate.

The Information Metric, on the other hand, is, as should be clear from what has been said above, very different. In fact, since the instanton density is invariant under all gauge transformations, the Information Metric on $\mathcal{M}_{k=1}(\mathbb{R}^4, SU(N))$ is completely degenerate along the $SU(N)$ gauge moduli directions and thus reduces to the $k = 1$ $SU(2)$ Information Metric on $\mathcal{M}_{k=1}(\mathbb{R}^4, SU(2))$, i.e. the AdS_5 -metric, for any N .

While this is an intriguing result, in particular in light of the AdS/CFT emergence of AdS_5 in the large- N instanton calculus [6], we do not quite know what to make of this.

2.3 THE GROISSER-MURRAY THEOREM AND THE RELATION TO THE FEFFERMAN-GRAHAM CONSTRUCTION

As shown above, it follows from the conformal invariance of the Information Metric that on $\mathcal{M}_{k=1}(S^4, SU(2))$ it coincides (up to a scale) with the standard hyperbolic (Euclidean AdS) metric on the five-ball.

Groisser and Murray have shown [10] that this is prototypical for the asymptotic behaviour of G_{AB} on the $k = 1$ $SU(2)$ moduli space for a much larger class of manifolds, in particular for S^4 (or \mathbb{R}^4) equipped with a metric which is not the standard (round or flat) metric.

Intuitively one would expect nearly point-like instantons on such a space to probe the geometry of the space-time X and its metric g_X . And the main result of Groisser and Murray is that indeed, as a suitably defined scale function $\rho \rightarrow 0$, the Information Metric G_{AB} on the 5-dimensional moduli space \mathcal{M} tends to

$$ds^2 \sim \frac{d\rho^2 + g_X}{\rho^2} . \quad (2.14)$$

In particular, therefore, for sufficiently small ρ the Information Metric is non-degenerate and asymptotically hyperbolic.

The encouraging thing about this result is that this is precisely the asymptotic form of the metric entering the Fefferman-Graham construction [14] and expected from the AdS/CFT correspondence.

Recall that in the Fefferman-Graham construction one makes an ansatz for the bulk metric G_{AB} which is to satisfy the Einstein equations with a negative cosmological constant, and with boundary value g_X , of the form

$$ds^2 = \frac{d\rho^2 + g_X(\rho)}{\rho^2} , \quad (2.15)$$

where

$$g_X(\rho) = g_X + \rho^2 g_X^{(2)} + \rho^4 g_X^{(4)} + \dots . \quad (2.16)$$

For $\rho \rightarrow 0$ this agrees precisely with the $\rho \rightarrow 0$ behaviour of the Information Metric. This means that the asymptotic behaviour of the metric is compatible with the bulk Information Metric being Einstein.

Note that, even though we know that the Information Metric is conformally invariant, the asymptotic form of the Information Metric given above appears to depend on the metric g_X itself, not only on its conformal class $[g_X]$.

However, here yet another useful fact about asymptotically AdS spaces comes to the rescue, namely that Weyl transformations of the boundary metric can be induced by certain bulk diffeomorphisms. These have been discussed in detail in [15]. Thus Weyl invariance of the Information Metric is indeed compatible with the Groisser-Murray form of the asymptotic metric, as the Information Metrics for different conformal factors are all diffeomorphic.

Moreover, the analysis of [15] shows that the first non-trivial term $g_X^{(2)}$ in the Fefferman-Graham expansion is determined uniquely by conformal invariance alone and agrees with the result obtained by Henningson and Skenderis [16] by solving the bulk Einstein equations to that order.

This means that, were we to calculate the Information Metric in a small- ρ expansion (we will not), then to first non-trivial order the result would be guaranteed to be compatible with the Information Metric being Einstein. At higher orders, however, conformal invariance does not fix the coefficients $g_X^{(2n)}$ uniquely as there are non-trivial higher-derivative Weyl invariants like the Euler density and the square of the Weyl tensor.

2.4 OTHER ASPECTS OF THE INFORMATION GEOMETRY OF \mathcal{M}_0

In this section we will analyse and summarise some aspects of the geometry of \mathcal{M}_0 equipped with the Information Metric. Of course the geometry of the five-dimensional hyperbolic plane or Euclidean AdS space is well known, and it is not our intention to review these facts here.

Rather, the purpose of this section is to focus on and highlight those aspects of the AdS/information geometry which we will show survive to first order in the metric perturbation. In particular, therefore, the (super-)symmetries of AdS will play no role in the following. In passing we will also list some identities which will be useful for the calculations of section 3.

One other aspect of the geometry of AdS, namely the boundary-to-bulk geodesic distance functional, which will turn out to play a fundamental role in the following, will be dealt with separately in the next section.

Thus we consider the standard AdS metric

$$ds_0^2 \equiv G_{AB} dy^A dy^B = \frac{d\rho^2 + d\vec{a}^2}{\rho^2} , \quad (2.17)$$

which, as we know, has the integral representation

$$G_{AB} = \frac{15}{8\pi^2} \int d^4x F^2 \partial_A \log F^2 \partial_B \log F^2 , \quad (2.18)$$

where F^2 was defined in (2.5). This metric is maximally symmetric, i.e. its curvature tensor is

$$R_{ABCD} = -(G_{AC}G_{BD} - G_{AD}G_{BC}) \quad (2.19)$$

and, in particular, Einstein,

$$R_{AB} = -4G_{AB} . \quad (2.20)$$

The function (actually four-parameter family of functions) $F^2(x; \rho, a^\mu)$ on \mathcal{M}_0 satisfies two interesting identities. Let ∇_A denote the AdS covariant derivative and \square the scalar Laplacian,

$$\square = G^{AB} \nabla_A \nabla_B = \rho^2 [\partial_\rho \partial_\rho + \delta^{\mu\nu} \partial_{a^\mu} \partial_{a^\nu} - 3\rho^{-1} \partial_\rho] . \quad (2.21)$$

Then the first identity is

$$\square F^2 = 0 . \quad (2.22)$$

As

$$\lim_{\rho \rightarrow 0} \frac{6}{\pi^2} F^2(x; \rho, a^\mu) = \delta(a - x) , \quad (2.23)$$

this establishes that $(6/\pi^2)F^2$ is the *boundary-to-bulk propagator* of a massless scalar field on AdS_5 [3]. I.e. for an arbitrary boundary scalar field $\phi(x)$, the bulk scalar $\Phi(\rho, a^\mu)$ defined by

$$\Phi(\rho, a^\mu) = \frac{6}{\pi^2} \int d^4x F^2(x; \rho, a^\mu) \phi(x) \quad (2.24)$$

satisfies

$$\square \Phi(\rho, a^\mu) = 0 \quad (2.25)$$

and

$$\lim_{\rho \rightarrow 0} \Phi(\rho, a^\mu) = \phi(a^\mu) . \quad (2.26)$$

It is also true that the massive scalar propagator is given by some power of F^2 . Indeed it can be verified that

$$\square (F^2)^{\Delta/4} = \Delta(\Delta - 4)(F^2)^{\Delta/4} . \quad (2.27)$$

In particular, $(F^2)^{1/2}$ corresponds to the tachyonic propagator with mass $m^2 = -4$. The identity (2.27) for all Δ follows from (2.22) and the rather remarkable identity

$$G^{AB}(y) \partial_A \log F^2(x; y) \partial_B \log F^2(x; y) = 16 \quad \forall x, y , \quad (2.28)$$

which can be readily verified by explicit calculation using the expressions for the logarithmic derivatives of F^2 given in (2.8).

These logarithmic derivatives of F^2 will be ubiquitous in the following, and it will be convenient to think of them as a four-parameter family (indexed by x) of covector fields on \mathcal{M}_0 we will call (with a convenient normalization)

$$v_A(x) := \frac{1}{4} \partial_A \log F^2 . \quad (2.29)$$

Obviously these vector fields satisfy

$$G^{AB} v_A(x) v_B(x) = 1 \quad \forall x \quad (2.30)$$

and

$$\nabla_A v_B = \nabla_B v_A \quad (2.31)$$

Some of their other main properties are

$$\nabla_A v_B = -G_{AB} + v_A v_B \quad (2.32)$$

and

$$v^A \nabla_B v_A = 0 \quad (2.33)$$

$$v^A \nabla_A v_B = 0 \quad (2.34)$$

$$G^{AB} \nabla_A v_B = -4 \ . \quad (2.35)$$

Property (2.33) follows from (2.30) by differentiation; (2.34) says that v_A is a geodesic vector field and follows from (2.31, 2.33); lastly, given (2.28), (2.35) is equivalent to (2.22).

Geometrically (2.30) means that we have a four-parameter family of constant norm or geodesic gradient vector fields on \mathcal{M}_0 . It is easy to see that these span the tangent space at each point of the moduli space.

In the following we will refer to (2.22) and (2.30) as the Propagator Equation and the Hamilton-Jacobi (HJ) Equation respectively (the reason for calling (2.30) the HJ equation will emerge below). We will show that these equations, as well as the Einstein equation, continue to be valid to first order in the metric perturbation, and we will study the resulting geometry imposed on \mathcal{M}_0 and its perturbation \mathcal{M} in much greater detail in section 4. In particular, the power of these identities derives from the fact that they hold for all x . Thus new identities on \mathcal{M} can be derived from them by differentiation with respect to x .

Property (2.32), on the other hand, which immediately implies (2.33)-(2.35), also implies that the metric G_{AB} is maximally symmetric and can thus not hold in general. Indeed, differentiating once more and taking commutators, one finds

$$[\nabla_C, \nabla_A]v_B = (G_{AB}G_{CD} - G_{BC}G_{AD})v^D \ . \quad (2.36)$$

Since this holds for all x , and the $v_A(x)$ span the tangent space of \mathcal{M}_0 , this implies that the curvature tensor of the bulk Information Metric is

$$R_{BDAC} = -(G_{AB}G_{CD} - G_{BC}G_{AD}) \ . \quad (2.37)$$

The identity (2.32) has one unobvious consequence that will play a role in the following, namely that

$$\int d^4x F^2(x) v_A(x) v_B(x) v_C(x) = 0 \quad \forall A, B, C \ . \quad (2.38)$$

To prove this, we use the fact that the Information Metric can be written in integral form as

$$G_{AB} \sim \int F^2 v_A v_B . \quad (2.39)$$

Since by definition

$$\nabla_C G_{AB} = 0 , \quad (2.40)$$

it follows that

$$\nabla_C \int F^2 v_A v_B = 0 . \quad (2.41)$$

Using

$$\int F^2 v_A = \frac{1}{4} \int \partial_A F^2 = \frac{1}{4} \partial_A \int F^2 = 0 \quad (2.42)$$

and (2.32), (2.38) follows.

One more identity that we will need is the ‘two-point’ version of (2.30). Namely, since $v^A(x)v_A(x) = 1$, it must be true that $v^A(x)v_A(y) - 1$ is proportional to $|x - y|$. The precise formula is

$$v^A(x)v_A(y) - 1 = \frac{2\rho^2(x - y)^2}{[(x - a)^2 + \rho^2][(y - a)^2 + \rho^2]} . \quad (2.43)$$

2.5 THE BOUNDARY-TO-BULK GEODESIC DISTANCE IN ADS AND THE INSTANTON DENSITY

The aim of this section is to show that

$$-\frac{1}{4} \log F^2 = \log \frac{(a - x)^2 + \rho^2}{\rho} \quad (2.44)$$

can be interpreted as the regularized geodesic distance in AdS between the bulk point (ρ, a) and the boundary point $(0, x)$ (regularized because the geodesic distance to the boundary at $\rho = 0$ is infinite). This provides an explanation for the validity of (2.30) as it can now be interpreted as the Hamilton-Jacobi equation for the classical geodesic action functional, as we will explain in section 2.6.

There are a number of different ways to establish this result. One is to integrate the geodesic equation directly. Alternatively one can use the *chordal distance* L of AdS_{d+1} , i.e. the restriction to

$$\eta_{AB} X^A X^B \equiv (X^1)^2 + \dots + (X^{d+1})^2 - (X^0)^2 = -1 . \quad (2.45)$$

of the unique $SO(d + 1, 1)$ -invariant distance

$$L(X_0, X_1) := \eta_{AB} (X_1^A - X_0^A)(X_1^B - X_0^B) \quad (2.46)$$

on the embedding space. A straightforward calculation shows that in Poincaré coordinates (ρ, a^μ) one has

$$L(\rho_0, a_0; \rho_1, a_1) = \frac{(\rho_1 - \rho_0)^2 + (a_1 - a_0)^2}{\rho_0 \rho_1} . \quad (2.47)$$

Because of the isometries of AdS_{d+1} , the geodesic distance function D can only be a universal function of the chordal distance L . Thus to determine this universal function, it is sufficient to consider purely radial geodesics ($a_1 = a_0$) for which evidently the geodesic distance is

$$D(\rho_0, a; \rho_1, a) = \log \frac{\rho_1}{\rho_0} . \quad (2.48)$$

This is related to the chordal distance

$$L(\rho_0, a; \rho_1, a) = \frac{\rho_1}{\rho_0} + \frac{\rho_0}{\rho_1} - 2 \quad (2.49)$$

by

$$L + 2 = e^D + e^{-D} = 2 \cosh D \quad (2.50)$$

or

$$L = 4 \sinh^2(D/2) , \quad (2.51)$$

which is thus the general relation between the chordal and geodesic distance. For L very large, the relation becomes

$$D \approx \log L . \quad (2.52)$$

In particular, therefore,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} D(\epsilon, x; \rho, a) &= \log \frac{\rho^2 + (a - x)^2}{\rho \epsilon} \\ &= \log \frac{\rho^2 + (a - x)^2}{\rho} - \log \epsilon , \end{aligned} \quad (2.53)$$

as was to be shown.

While this is a quick and clean argument, it has the disadvantage of relying on the isometries of AdS, and can therefore not generalize to the perturbed instanton moduli spaces we are interested in. We will now give an argument that, as we will see later, does generalize since it only relies on the validity of the HJ equation (2.30).

We know from (2.34) that the v_A are geodesic vector fields,

$$v^A \nabla_A v_B = 0 . \quad (2.54)$$

Let $y^A(\tau) = (\rho(\tau), a(\tau))$ be the corresponding geodesic curves. Thus we have

$$\dot{y}^A = G^{AB} v_B . \quad (2.55)$$

To integrate this equation, we calculate

$$\frac{d}{d\tau} \log F^2 = \dot{y}^A \partial_A \log F^2 = 4v^A v_A = 4 \quad , \quad (2.56)$$

so that these geodesic curves can be simply described by the equation

$$\log F^2(x; y(\tau)) = \log F^2(x; y(\tau = 0)) + 4\tau \quad . \quad (2.57)$$

Thus the proper distance is

$$D = \int_{\tau=0}^{\tau=\tau_+} d\tau = \frac{1}{4} \log F^2(x; y(\tau_+)) - \frac{1}{4} \log F^2(x; y(\tau = 0)) \quad . \quad (2.58)$$

We choose $y^A(0) = (a_0, \rho_0)$ and $y^A(\tau_+) = (\epsilon, a_+)$ and note that for $\epsilon \rightarrow 0$ we necessarily have $\tau \rightarrow \infty$. It remains to show that as $\epsilon \rightarrow 0$ we also have $a_+ = x$. This follows from the fact that

$$\lim_{\epsilon \rightarrow 0} \log F^2(x; \epsilon, a) = \pm \infty \quad , \quad (2.59)$$

the upper sign occuring if $x = a$, and the lower if $x \neq a$. As the geodesic distance is a positive quantity (the v^A point towards the boundary and we are integrating towards the boundary), we must have the upper sign and hence $x = a_+$. Regularizing the geodesic distance as before by removing this infinity, we reproduce precisely the result (2.53).

2.6 $G^{AB}v_A v_B = 1$ IS THE HAMILTON-JACOBI EQUATION FOR GEODESIC MOTION ON \mathcal{M}_0

Since we have shown that $D = -\frac{1}{4} \log F^2 - \log \epsilon$ is the geodesic distance, it must satisfy the Hamilton-Jacobi equation (Hamiltonian constraint)

$$G^{AB} p_A p_B = 1 \quad , \quad (2.60)$$

where the canonical momentum p_A is obtained by varying the classical geodesic action $S_{cl}(y) = D(\epsilon, x; y)$ with respect to the end-point $y = (\rho, a)$,

$$p_A = \partial_A D(\epsilon, x; y) \quad (2.61)$$

In particular, the p_A are finite as $\epsilon \rightarrow 0$. Thus the Hamilton-Jacobi equation for boundary-to-bulk paths is

$$G^{AB} \partial_A D(0, x; y) \partial_B D(0, x; y) = 1 \quad , \quad (2.62)$$

which is nothing other than the equation $G^{AB} v_A(x) v_B(x) = 1$.

We will show later that the relation between the geodesic distance and the instanton density continues to hold to first order in the metric perturbation, and this may provide

a rationale for the validity of the (in other respects somewhat mysterious) HJ equation to that order.

Before leaving this topic we would like to point out that these results suggest that a WKB type approximation to the path integral for the boundary-to-bulk scalar propagator should be exact.

3 FIRST ORDER CORRECTIONS

In this section we will determine the first order corrections to the Information metric, massless scalar boundary-to-bulk propagator and the geodesic distance (Hamilton Jacobi equation) when one deforms the 4-dimensional space on which the Yang-Mills theory lives. The first step in this computation therefore is to calculate the correction to the instanton solution to first order in the perturbation of the boundary space-time metric.

3.1 METRIC DEFORMATIONS AND INSTANTON DEFORMATIONS

To determine the change in the Information Metric on the instanton moduli space as we vary the metric on S^4 requires knowledge of the variation of the instanton density

$$\text{tr } F_A * F_A = \text{tr } F_A \wedge F_A \quad , \quad (3.1)$$

i.e.

$$\delta_h \text{tr}(F_A \wedge F_A) = 2 \text{tr}(F_A \wedge \delta_h F_A) \quad (3.2)$$

where δ_h represents the variation $\delta_h g_{\mu\nu} = h_{\mu\nu}$ of the boundary metric. Notice that since F_A is self-dual only the self-dual part of the variation $\delta_h F_A$ enters on the right hand side. However, the deformation of the self-duality equations gives an algebraic equation only for the anti-self-dual part of $\delta_h F_A$, see (3.5) below. This means that we need to get an explicit formula for the deformed instanton so as to be able to determine (3.2).

The ideas here go back to the work of Taubes [17] on the ‘grafting’ of instantons from S^4 to an arbitrary 4-manifold. In our case we are grafting the instantons on S^4 to an S^4 which is equipped with a different metric. Let $*_0$ be the Hodge star operator for the round metric on S^4 and let $* = *_0 + \delta_h *$ denote the perturbed Hodge operator. $\delta_h * = t *_1 + t^2 *_2 + \dots$. Here we work to first order in t . In local coordinates we write the metric as $g_{\mu\nu} = g_{\mu\nu}^0 + t h_{\mu\nu} + \dots$. While the above refers to S^4 all of our formulae are written in the context of \mathbb{R}^4 .

We wish to solve the equation

$$F_A = *F_A. \quad (3.3)$$

Write the connection as

$$A = A_0 + t\omega + \dots \quad (3.4)$$

where A_0 is the instanton solution on the round S^4 . Differentiating (3.3) once with respect to t and setting $t = 0$ we obtain

$$(1 - *_0)d_{A_0}\omega = *_1 F_{A_0}, \quad (3.5)$$

so that, in particular, $*_1 F_{A_0}$ is anti-self-dual.

The choice of ω is dictated by the requirements that it is perpendicular to gauge transformations and also does not represent a tangent vector to the moduli space \mathcal{M} . One takes

$$\omega = *_0 d_{A_0} u \quad (3.6)$$

where $u = \frac{1}{2}u_{\mu\nu}dx^\mu dx^\nu$ is an anti-self-dual two-form. With this choice (3.5) becomes an elliptic equation,

$$\Delta_0 u = *_1 F_{A_0} \quad , \quad (3.7)$$

where

$$\Delta_0 = -(d_{A_0} *_0 d_{A_0} *_0 + *_0 d_{A_0} *_0 d_{A_0}) \quad (3.8)$$

is the Laplacian on two-forms, so that one can solve for u . In local coordinates, (3.6) is

$$\omega_\mu = -D_\lambda^0 \omega_\mu^\lambda \quad , \quad (3.9)$$

and the Green's function form of (3.7) reads

$$D_0^2 u_{\mu\nu}(x, y) = -\delta(x - y) (*_1 F_{A_0})_{\mu\nu}(y), \quad (3.10)$$

where

$$(*_1 F_{A_0})_{\mu\nu} = h_\mu^\alpha F_{\alpha\nu}(A_0) - h_\nu^\alpha F_{\alpha\mu}(A_0). \quad (3.11)$$

Here labels are raised with the original unperturbed metric. Notice also that there is no term $h_\alpha^\alpha F_{\mu\nu}(A_0)$ as the instanton equation is conformally invariant. For this reason we will henceforth only consider traceless perturbations $h_{\mu\nu}$. Also, as there is no likelihood of confusion we will from now on, and until section 4, drop the zero subscript.

One can solve the Green's function equation [18], with the result that

$$u_{\mu\nu}(x, y) = \frac{\overline{U}(x)U(y)(*_1 F)_{\mu\nu}(y)\overline{U}(y)U(x)}{4\pi^2(x - y)^2}, \quad (3.12)$$

where U is a 4×2 matrix in terms of which the self-dual gauge potential (at zero'th order) is simply $A_\mu = i\bar{U}\partial_\mu U$. Explicitly U is

$$U(x) = \frac{1}{\sqrt{(x - a)^2 + \rho^2}} \begin{pmatrix} (x - a)^\mu \bar{\sigma}_\mu \\ -\rho \end{pmatrix} g(x) \quad (3.13)$$

and $g(x)$ is an unspecified $SU(2)$ gauge group element. Since we will be dealing with gauge invariant quantities there is no reason to fix on a particular $g(x)$.

The σ -matrices are defined by

$$\sigma_\mu = (\mathbb{I}, i\tau_a) \ , \quad \bar{\sigma}_\mu = (\mathbb{I}, -i\tau_a) \ , \quad (3.14)$$

where the τ_a are the standard Pauli matrices. They satisfy

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\delta_{\mu\nu} \quad (3.15)$$

and are such that

$$2\sigma_{\mu\nu} \equiv \sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu = 2i\eta_{\mu\nu}^a \tau_a \quad (3.16)$$

is self-dual. One has the identities

$$\sigma_\mu^\lambda A_{\nu\lambda}^{(-)} - \sigma_\nu^\lambda A_{\mu\lambda}^{(-)} = 0 \ , \quad \sigma_\mu^\lambda A_{\nu\lambda}^{(+)} + \sigma_\nu^\lambda A_{\mu\lambda}^{(+)} = \frac{1}{4}\delta_{\mu\nu}\sigma^{\alpha\beta} A_{\alpha\beta}^{(+)} \ , \quad (3.17)$$

for any anti-self-dual and self-dual tensors $A^{(-)}$ and $A^{(+)}$ respectively.

Another useful fact that is often used in computations is that to first order

$$\begin{aligned} \delta_h \operatorname{tr} F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} &= 4 \operatorname{tr} F_{\mu\nu} \delta^{\mu\alpha} \delta^{\nu\beta} D_\alpha \omega_\beta \\ &= 4\partial^\mu \operatorname{tr} F_{\mu\nu} \omega^\nu \\ &= 4\partial^\mu \partial^\alpha \operatorname{tr} F_{\mu\nu} u^\nu{}_\alpha \ . \end{aligned} \quad (3.18)$$

The first equality follows since if we change the metric the inner product $\operatorname{tr} F_A *_1 F_A = 0$, the second follows by the Bianchi identity (this is the usual relationship $\delta_h \operatorname{tr} F_A^2 = 2d \operatorname{tr} F_A \delta_h A$), the third by the Bianchi identity again.

3.2 THE LINEARIZED BULK EINSTEIN EQUATION

We will now compute the first order deformation of the Information metric (2.11) arising from the deformation of the flat metric on \mathbb{R}^4 via the first order correction to the instanton solution.

From the definition of the Information metric it follows that

$$\delta_h G_{AB} = \delta_h G'_{AB} + \nabla_{(A} V_{B)} \ , \quad (3.19)$$

where

$$\delta_h G'_{AB} = -\frac{5}{64\pi^2} \int d^4x \ (\delta_h \operatorname{tr} F^2) \frac{\nabla_A \nabla_B \sqrt{\operatorname{tr} F^2}}{\sqrt{\operatorname{tr} F^2}} \quad (3.20)$$

and

$$V_A = \frac{5}{64\pi^2} \int d^4x \ (\delta_h \operatorname{tr} F^2) \partial_A \ln \operatorname{tr} F^2 \ . \quad (3.21)$$

Here we have used the fact that F^2 as defined in (2.5) is normalized as $1/96$ times $\text{tr } F^2$ defined in (2.4). The covariant derivatives ∇_A are with respect to the unperturbed AdS_5 metric. $\delta_h G$ and $\delta_h G'$ are therefore related by a bulk diffeomorphism.

We will see below that, somewhat unexpectedly but very conveniently, $\delta_h G'_{AB}$ satisfies the transverse-traceless gauge condition. Therefore the linearized Einstein equation is simply

$$\square \delta_h G'_{AB} = (\Lambda/2) \delta_h G'_{AB} \quad , \quad (3.22)$$

when the unperturbed metric satisfies $R_{AB} = \Lambda G_{AB}$.

$\delta_h G'_{AB}$ can be simplified further by using (2.29) and (2.32) to

$$\delta_h G'_{AB} = -\frac{15}{32\pi^2} \int d^4x \, v_A v_B \delta_h \text{tr } F^2 \quad , \quad (3.23)$$

where we have used the fact that $\int d^4x \, \delta_h \text{tr } F^2 = 0$.

Using $v^A v_A = 1$ it follows that $\delta_h G'_{AB}$ is traceless, $G^{AB} \delta_h G'_{AB} = 0$. Furthermore, using (2.34) and (2.35) one finds

$$\begin{aligned} \nabla^A \delta_h G'_{AB} &= -\frac{15}{32\pi^2} \int d^4x \, v_B (v^A \nabla_A - 4) \delta_h \text{tr } F^2 \\ &= -\frac{15}{8\pi^2} \int d^4x \, F^2 v_B v^A \delta_h v_A \quad . \end{aligned} \quad (3.24)$$

In the next subsection we will establish that the HJ equation holds to first order in the metric variation, i.e. that

$$2v^A(x) \delta_h v_A(x) = -v_A(x) v_C(x) \delta_h G^{AC} \quad . \quad (3.25)$$

Substituting this in (3.24) and using (2.38), it follows that $\delta_h G'_{AB}$ is also transverse.

To proceed further we need the expression for $\delta_h \text{tr } F^2$. Substituting the explicit expressions for $u_{\mu\nu}$ as given by (3.12) and (3.13) in (3.18) we have

$$\delta_h \text{tr } F^2(x) = \partial_\mu \partial_\nu S_{\mu\nu} \quad (3.26)$$

with

$$S_{\mu\nu} = -\frac{4}{\pi^2 \rho^2} \int d^4y (F^2(x) F^2(y))^{\frac{3}{4}} \frac{1}{(x-y)^2} \text{tr}(\rho^2 + Y \bar{X}) \sigma_{\mu\lambda} (\rho^2 + X \bar{Y}) H_{\lambda\nu} \quad . \quad (3.27)$$

Here capital X and Y denote $(x-a)$ and $(y-a)$ respectively, $\bar{X} = X^\mu \bar{\sigma}_\mu$, a and ρ are the position and scale of the unperturbed instanton, and

$$H_{\lambda\nu}(y) = h_\lambda{}^\mu(y) \sigma_{\mu\nu} - h_\nu{}^\mu(y) \sigma_{\mu\lambda} \quad (3.28)$$

is anti-self-dual. We have set $h_\mu^\mu = 0$ since this does not change the Information Metric due to the conformal invariance.

We sketch below the computation for $\delta_h G'_{\rho\rho}$. The remaining components of the metric deformation as well as the diffeomorphism vector field V_A can be computed in a similar way.

Making use of the σ -matrix identities and the fact that $v_\rho = (X^2 - \rho^2)/\rho(X^2 + \rho^2)$, we find that

$$\delta_h G'_{\rho\rho} = -\frac{240}{\pi^4} \int d^4 y \int d^4 x \frac{\rho^4 (X^2 - 2\rho^2)(\rho^2 X^\mu + X^2 Y^\mu)(\rho^2 X^\nu + X^2 Y^\nu)}{(Y^2 + \rho^2)^3 (X^2 + \rho^2)^7 (x - y)^2} h_{\mu\nu}(y). \quad (3.29)$$

By writing $X^2 = (X^2 + \rho^2) - \rho^2$ in the numerator we can bring the x integral in the form of sums of terms like $f(X)/(X - Y)^2 (X^2 + \rho^2)^n$ where $f(X)$ is a quadratic polynomial of the form $B_{\mu\nu} X^\mu X^\nu + B_\mu X^\mu + B$. The x -integral of such terms can be performed by using Feynman parametrization

$$\frac{1}{(X^2 + \rho^2)^n (X - Y)^2} = \int_1^\infty dt \frac{n(t-1)^{n-1} t^{n+1}}{[(tX - Y)^2 + (t-1)(Y^2 + t\rho^2)]^{n+1}} , \quad (3.30)$$

shifting $X \rightarrow (X + Y)/t$, and using the formula

$$\int d^4 X \frac{1}{(X^2 + A)^n} = \frac{\pi^2}{(n-1)(n-2)} \frac{1}{A^{n-2}} . \quad (3.31)$$

Finally the t -integral is easily computed with the result

$$\delta_h G'_{\rho\rho} = \frac{40}{\pi^2} \int d^4 y \frac{\rho^4}{(Y^2 + \rho^2)^6} Y^\mu Y^\nu h_{\mu\nu}(y) . \quad (3.32)$$

Similarly we can compute the other components of the metric deformation in the transverse traceless gauge and the result is

$$\begin{aligned} \delta_h G'_{\rho a} &= \frac{20}{\pi^2} \int d^4 y \frac{\rho^3}{(Y^2 + \rho^2)^5} [Y^\mu h_{\mu a}(y) - 2 \frac{1}{Y^2 + \rho^2} Y^\mu Y^\nu h_{\mu\nu}(y) Y_a] \\ \delta_h G'_{ab} &= \frac{10}{\pi^2} \int d^4 y \frac{\rho^2}{(Y^2 + \rho^2)^4} [h_{ab}(y) - 2 \frac{1}{Y^2 + \rho^2} Y^\mu h_{\mu(a}(y) Y_{b)} + \\ &\quad 4 \frac{1}{(Y^2 + \rho^2)^2} Y^\mu Y^\nu h_{\mu\nu}(y) Y_a Y_b] . \end{aligned} \quad (3.33)$$

Note that in the $\rho \rightarrow 0$ limit, the leading term in the metric deformation is in the Feffermann-Graham form, namely,

$$\delta_h G'_{\rho\rho} \rightarrow 0 , \quad \delta_h G'_{\rho a} \rightarrow 0 , \quad \delta_h G'_{ab} \rightarrow \frac{1}{\rho^2} h_{ab} , \quad (3.34)$$

thus also in agreement with the predictions of the Groisser-Murray theorem discussed in section 2.3.

Having obtained $\delta_h G'_{AB}$, we could now attempt to verify the linearized Einstein equation (3.22) (with $\Lambda = -4$) by a direct calculation. However, we can side-step this calculation by noting that the expressions (3.32) and (3.33) coincide exactly with the AdS_5 boundary-to-bulk graviton propagator in the transverse traceless gauge obtained in [19]! This proves that to $\mathcal{O}(h)$, i.e. to first order in the metric perturbation, the deformation of the Information Metric satisfies the bulk Einstein equation.

It is instructive (and necessary for the subsequent calculations) to compute $\delta_h G_{AB}$ which appears naturally in the definition of the Information Metric. For this we need to compute the diffeomorphism vector field V_A . These integrals can be performed using the above steps. The result is

$$\begin{aligned} V_\rho &= \frac{1}{\pi^2} \int d^4 y \frac{\rho}{(Y^2 + \rho^2)^4} [1 + \frac{2\rho^2}{Y^2 + \rho^2}] Y^\mu Y^\nu h_{\mu\nu}(y) \\ V_a &= \frac{2}{\pi^2} \int d^4 y \frac{\rho^2}{(Y^2 + \rho^2)^4} [\delta_a^\nu - \frac{Y^\nu Y_a}{Y^2 + \rho^2}] Y^\mu h_{\mu\nu}(y) . \end{aligned} \quad (3.35)$$

Substituting V_A and $\delta_h G'_{AB}$ in (3.19) we finally obtain the following expressions for $\delta_h G_{AB}$:

$$\begin{aligned} \delta_h G_{\rho\rho} &= \frac{4}{\pi^2} \int d^4 y \frac{1}{(Y^2 + \rho^2)^4} Y^\mu Y^\nu h_{\mu\nu}(y) \\ \delta_h G_{\rho a} &= \frac{6}{\pi^2} \int d^4 y \frac{\rho}{(Y^2 + \rho^2)^4} Y^\mu h_{\mu a}(y) \\ \delta_h G_{ab} &= \frac{6}{\pi^2} \int d^4 y \frac{1}{(Y^2 + \rho^2)^4} [\rho^2 h_{ab}(y) - \frac{1}{3} \delta_{ab} Y^\mu Y^\nu h_{\mu\nu}(y)] . \end{aligned} \quad (3.36)$$

δG_{AB} also (and somewhat more manifestly) satisfies (3.34).

It is natural to ask what happens if the metric variation $h_{\mu\nu}$ is simply a diffeomorphism,

$$h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu , \quad (3.37)$$

which acts non-trivially on the instanton density but which should not change the bulk geometry. Indeed, by plugging (3.37) into (3.36), one finds that

$$\delta_h G_{AB} = \nabla_A X_B + \nabla_B X_A \quad (3.38)$$

is a bulk diffeomorphism, with $X_\rho = 0$ and

$$X_a(\rho, a) = \frac{6}{\pi^2} \int d^4 y \frac{\rho^2}{(Y^2 + \rho^2)^4} \xi_a(y) . \quad (3.39)$$

This is the natural lift of ξ_μ to \mathcal{M}_0 with

$$\lim_{\rho \rightarrow 0} X_a(\rho, a) = \frac{1}{\rho^2} \xi_a(a) . \quad (3.40)$$

3.3 THE HAMILTON-JACOBI EQUATION

We now show that to the first order in the metric perturbation, the Hamilton-Jacobi equation (2.30) which, as we have seen, says that $\text{tr } F^2$ is related to the (regularized) geodesic boundary-to-bulk geodesic distance, still holds. Taking the variation of (2.30), it follows that we need to prove

$$4F^2(x)v_A(x)G^{AB}\delta_h v_B(x) = -2F^2(x)v_A(x)v_B(x)\delta_h G^{AB} \quad (3.41)$$

where we have multiplied both sides by $2F^2(x)$ for convenience.

This equation depends on x and the moduli as well as the perturbation $h_{\mu\nu}(y)$. Since the perturbation is arbitrary (with suitable falloff conditions at large y), the above equation must be true for each point y . Therefore we will drop the y -integrals in (3.27) and (3.36) in the following. Note that $\delta_h F^2$, appearing on the left hand side, has a singularity as x approaches y while the right hand side has no such singularity. Thus the singularities on the left hand side must cancel if the above equality is to hold. We will see below that this is indeed the case.

First let us evaluate the right hand side. Using the expressions (3.36) for $\delta_h G_{AB}$ one easily finds that

$$\begin{aligned} -2F^2(x)v_A(x)v_B(x)\delta_h G^{AB} &= \frac{48}{\pi^2\rho^4}(F^2(x))^{\frac{3}{2}}F^2(y)[\rho^4 X^\mu X^\nu + \rho^2 X^\mu Y^\nu(X^2 - \rho^2) + \\ &\quad \frac{1}{6}Y^\mu Y^\nu(X^4 - 4X^2\rho^2 + \rho^4)]h_{\mu\nu}(y) \end{aligned} \quad (3.42)$$

Now we proceed to compute the left hand side of (3.41). Using the form of $\delta_h \text{tr } F^2$ (3.26, 3.27), the left hand side can be written as

$$\begin{aligned} 4F^2(x)v_A(x)G^{AB}\delta_h v_B(x) &= \frac{1}{96}(v^A\partial_A - 4)\delta_h \text{tr } F^2 \\ &= \frac{1}{96}(\partial_\mu\partial_\nu T^{\mu\nu} - 2\partial_\mu T^\mu + T) \end{aligned} \quad (3.43)$$

where

$$\begin{aligned} T^{\mu\nu} &= (v^A(x)\partial_A - 4)S^{\mu\nu} \\ T^\mu &= v_\nu^A(x)\partial_A S^{\mu\nu} \\ T &= v_{\mu\nu}^A(x)\partial_A S^{\mu\nu} \end{aligned} \quad (3.44)$$

and $S^{\mu\nu}$ was defined in (3.27). Here v_μ^A and $v_{\mu\nu}^A$ indicate derivatives of v^A with respect to x^μ , x^ν etc. and the moduli space indices A, B are raised and lowered with the AdS_5 metric G_{AB} . The 5-dimensional tangent vector space at each point on AdS_5 is spanned

by the 5 vectors v^A and v_μ^A (we will discuss this in more detail in section 4.2). Therefore $v_{\mu\nu}^A$ can be expressed as a linear combination of v^A and v_λ^A . A short calculation shows that

$$v_{\mu\nu}^A = -\delta_{\mu\nu}(4(F^2)^{1/2}v^A - z_\lambda v_\lambda^A) + (z_\mu v_\nu^A + z_\nu v_\mu^A) \quad (3.45)$$

where $z_\mu = \frac{1}{4}\partial_\mu \ln F^2(x)$. Using now the fact that $S_\mu^\mu = 0$, we obtain

$$T = 2z_\mu T^\mu \quad (3.46)$$

$T^{\mu\nu}$ and T^μ can be calculated explicitly using the expression (3.27) for $S^{\mu\nu}$. It is convenient to write $S_{\mu\nu}$ as

$$S_{\mu\nu} = -\frac{4}{\pi^2(x-y)^2}(F^2(x)F^2(y))^{\frac{3}{4}}\hat{S}_{\mu\nu} \quad (3.47)$$

with

$$\hat{S}^{\mu\nu} = \frac{1}{\rho^2} \text{tr}(\rho^2 + Y\bar{X})\sigma^{\mu\lambda}(\rho^2 + X\bar{Y})H_\lambda^\nu(y) \quad (3.48)$$

Then one can show that $\hat{S}_{\mu\nu}$ satisfies the following identities:

$$\begin{aligned} (v^A(x)\partial_A + 2)\hat{S}_{\mu\nu} &= (v^A(y)\partial_A + 2)\hat{S}_{\mu\nu} = 0 \\ (\square - 12)\hat{S}_{\mu\nu} &= 0 \end{aligned} \quad (3.49)$$

Using these identities it is straightforward to show that:

$$\begin{aligned} T^{\mu\nu} &= -\frac{4}{\pi^2(x-y)^2}(F^2(x)F^2(y))^{\frac{3}{4}}(v^A(x)\partial_A + 3v^A(x)(v_A(x) + v_A(y)) - 4)\hat{S}^{\mu\nu} \\ &= 6\frac{\rho^2(x-y)^2}{(\rho^2 + X^2)(\rho^2 + Y^2)}S^{\mu\nu} \end{aligned} \quad (3.50)$$

where we have used (3.49) and the explicit expression of $v^A(x)v_A(y)$ given in (2.43). The computation of T^μ is similar but considerably more involved. The result is:

$$T^\mu = \frac{1}{3}(\partial_\nu - z_\nu)T^{\mu\nu} \quad (3.51)$$

Combining the equations (3.43), (3.44), (3.46) and (3.51) we have

$$\begin{aligned} 4F^2v_A(x)G^{AB}\delta_h v_B(x) &= \frac{1}{288}(\partial_\mu + 4z_\mu)\partial_\nu T^{\mu\nu} \\ &= -\frac{1}{12\pi^2}F^2(x)F^2(y)(\partial_\mu + 8z_\mu)(\partial_\nu + 4z_\nu)\hat{S}^{\mu\nu} \end{aligned} \quad (3.52)$$

Using the identities (3.15, 3.16) involving σ -matrices and their contractions with the anti-self-dual tensor H given in (3.17), one can show that

$$\begin{aligned} \partial_\mu\partial_\nu\hat{S}^{\mu\nu} &= -\frac{96}{\rho^2}Y^\mu Y^\nu h_{\mu\nu}(y) \\ z_\mu\partial_\nu\hat{S}^{\mu\nu} &= \frac{48}{\rho^2(\rho^2 + X^2)}(X^2Y^\mu Y^\nu + \rho^2Y^\mu X^\nu)h_{\mu\nu}(y) \\ z_\mu z_\nu\hat{S}^{\mu\nu} &= -\frac{16}{\rho^2(\rho^2 + X^2)^2}(X^2Y^\mu + \rho^2X^\mu)h_{\mu\nu}(y)(X^2Y^\nu + \rho^2X^\nu) \end{aligned} \quad (3.53)$$

Plugging these expressions into (3.52), one finds after a short algebra that it reproduces the right hand side of (3.41) given by (3.42). We have thus established the validity of the HJ equation to $\mathcal{O}(h)$ in the perturbation of the boundary metric.

We also want to point out that the calculation of this section suggests an alternative strategy for determining the variation $\delta_h G_{AB}$ of the Information Metric. Namely, as the above calculation has shown, calculating $v^A \delta_h v_A$ one finds

$$2v^A \delta_h v_A = -v_A v_C M^{AC} , \quad (3.54)$$

where M^{AC} is an x -independent matrix. We can therefore define a variation of G_{AB} by, say, $\delta G^{AB} = M^{AB}$. This metric variation, by definition, preserves the HJ equation and will also preserve the Propagator Equation. If one proceeds in this way (which is simpler than the brute-force determination of $\delta_h G_{AB}$ of section 3.2), then one still has to show that indeed $M^{AB} = \delta_h G^{AB}$, i.e. that M^{AB} is what one would have obtained by varying the integral representation (2.1) of the Information Metric. We establish this in section 4.9.

3.4 THE MASSLESS BOUNDARY-TO-BULK SCALAR PROPAGATOR

We know that the boundary-to-bulk scalar propagator of AdS is simply $\text{tr } F^2$. We will now show that to first order in h , $\text{tr } F^2$ is still the boundary-to-bulk scalar propagator with respect to the bulk Information Metric. This amounts to proving

$$\square \delta_h \text{tr } F^2(x) + (\delta_h \square) \text{tr } F^2(x) = 0 \quad (3.55)$$

This equation depends on x and the moduli as well as on the perturbation $h_{\mu\nu}(y)$. As discussed above, this equation must therefore be true at each point x, y and at each point of the moduli space. Note that $\delta_h \text{tr } F^2$ in the first term on the left hand side has a $1/(x-y)^2$ pole while the second term has no such singularity. Thus for this equation to hold the action of the AdS Laplacian should remove this singularity.

Indeed, expanding x around y in the expression for $S_{\mu\nu}$ defined in (3.27), we find that the coefficient of the leading singularity $1/(x-y)^2$ is proportional to $F^2(y)$ which is annihilated by the AdS Laplacian. Similarly the first subleading singularity of the form $(x-y)^\mu/(x-y)^2$ comes with a coefficient which is given by a single y -derivative of $F^2(y)$ which is also annihilated by the AdS Laplacian.

Actually we need more than just the cancellation of singularities for the above equation to work. In the second term, the x -dependence only appears in $\text{tr } F^2$ while the y -dependence appears only in $\delta_h G$ implicit in $\delta_h \square$. This means that the second term is a

finite sum of factorized expressions in x and y . Quite remarkably, using the identities (3.49) satisfied by $\hat{S}_{\mu\nu}$ one can show that

$$\begin{aligned}\square S_{\mu\nu}(x, y) &= -\frac{4}{\pi^2}(F^2(x)F^2(y))^{\frac{3}{4}}(\square + 6(v^A(x) + v^A(y))\partial_A - 6 + 18v^A(x)v_A(y))\hat{S}_{\mu\nu} \\ &= 36\frac{\rho^2(x-y)^2}{(\rho^2 + X^2)(\rho^2 + Y^2)}S_{\mu\nu}(x, y)\end{aligned}\quad (3.56)$$

where in the first equality we have used (2.30) and (2.35) and in the second equality (3.49) and (2.43). Thus the first term also becomes a finite sum of factorized expressions in x and y . Using now the fact that

$$(\delta_h \square) \text{tr} F^2(x) = \frac{1}{\sqrt{G}}\partial_A \delta_h(G^{AB}\sqrt{G})\partial_B \text{tr} F^2 \quad (3.57)$$

and the metric variation (3.36), one can verify after some algebra that (3.55) is indeed true.

4 THE INFORMATION GEOMETRY OF THE PERTURBED INSTANTON MODULI SPACE \mathcal{M}

4.1 PRELIMINARY REMARKS

In the previous sections we established three key results about the Information Metric G_{AB} on the metric-perturbed instanton moduli space, namely that

1. the Einstein Equation

$$R_{AB} = -4G_{AB} \quad , \quad (4.1)$$

2. the Propagator Equation

$$\square F^2 = 0 \quad , \quad (4.2)$$

3. and the Hamilton-Jacobi Equation

$$G^{AB}v_A(x)v_B(x) = 1 \quad (4.3)$$

hold to $\mathcal{O}(h)$ in the perturbation of the metric on \mathbb{R}^4 .

These results also imply, as for the flat metric, that to $\mathcal{O}(h)$ the massive boundary-to-bulk scalar propagator is $(F^2)^{\Delta/4}$, with

$$\square(F^2)^{\Delta/4} = \Delta(\Delta - 4)(F^2)^{\Delta/4} \quad . \quad (4.4)$$

While all this is very encouraging, evidently a number of things remain to be understood. For example, so far we have no understanding to which extent these three properties are independent of or dependent on each other. We also do not yet know if we can expect these equations to remain valid at second or higher order in the metric perturbation.

To address these questions we will now explore the geometry of the instanton moduli space implied by the equations (4.1, 4.2, 4.3). We will see that it is in particular the (somewhat mysterious) HJ equation (4.3) which endows the moduli space with a very rich geometrical structure.

As a consequence we will be able to show that

- at first order in the metric perturbation the three fundamental equations are not independent: the HJ equation (4.3) and the propagator equation (4.2) imply the Einstein equation (4.1); and that
- these three equations cannot hold simultaneously to quadratic order in the metric perturbation.

Throughout we will see that the geometry of the Information Metric is interwoven in a subtle and beautiful way with the geometry of the boundary space-time, and we believe that understanding these structures will eventually lead to a better geometrical understanding of the metric variation of the instantons themselves.

4.2 ELEMENTARY CONSEQUENCES OF THE HAMILTON-JACOBI EQUATION

The HJ equation

$$G^{AB}v_A(x)v_B(x) = 1 \tag{4.5}$$

is quite remarkable as the right hand side is x -independent even though the x -dependent quantities $v_A(x)$ are contracted with the x -independent Information Metric G^{AB} .

Since for the flat metric on \mathbb{R}^4 , the vector fields $v_A(x)$ span the tangent space at each point of the moduli space \mathcal{M}_0 , they will continue to do so on the perturbed moduli space \mathcal{M} .

By differentiating the above equation with respect to x , we obtain

$$G^{AB}\partial_\mu v_A(x)v_B(x) = 0 \quad . \tag{4.6}$$

Hence the four gradient vector fields

$$v_{A\mu} \equiv \partial_\mu v_A = \partial_\mu \partial_A \log F^2 \tag{4.7}$$

on \mathcal{M} (labelled by μ) are orthogonal to v_A . Moreover, for the flat metric on \mathbb{R}^4 they are mutually orthogonal (super- or subscripts 0 will from now on always refer to the unperturbed instantons),

$$G_0^{AB} v_{A\mu}^0 v_{B\nu}^0 = 4\delta_{\mu\nu} (F_0^2)^{1/2} , \quad (4.8)$$

and thus they are also linearly independent on the perturbed moduli space. We will denote the non-degenerate scalar product of the $v_{A\mu}$ by $K_{\mu\nu}$,

$$K_{\mu\nu} = G^{AB} v_{A\mu} v_{B\nu} . \quad (4.9)$$

Note that

$$K_{\mu\nu}^0 = 4\delta_{\mu\nu} \frac{\rho^2}{[(x-a)^2 + \rho^2]^2} \quad (4.10)$$

is, up to the moduli-dependent coordinate transformation

$$x'^\mu = (x^\mu - a^\mu)/\rho , \quad (4.11)$$

simply the round metric (of radius 1) on the four-sphere in stereographic coordinates.

Starting from the HJ equation, we have therefore produced a four-parameter family of frames (five linearly independent vector fields) on \mathcal{M} . Actually, as all the vector fields are gradient vector fields, these correspond to a four-parameter family of coordinate systems on \mathcal{M} - we will come back to this aspect of the story later.

In any case it follows that the Information Metric on the moduli space can be written purely algebraically as

$$G_{AB} = v_A(x) v_B(x) + K^{\mu\nu}(x) v_{A\mu}(x) v_{B\nu}(x) \quad (4.12)$$

for any x . Once again it is a remarkable fact that the left hand side of this equation is completely independent of x .

Obviously, by construction, the integral and algebraic form of G_{AB} are equivalent if the metric entering the HJ equation (4.5) is the integral version of the Information Metric. On the other hand if, as proposed at the end of section 3.3, we define the variation of the Information Metric directly through the variation of the HJ equation, i.e. by (3.54), then we still need to show that the variation defined in this way is equal to the variation one would have obtained by varying the integral version of the Information Metric. In order to accomplish this, we need to accumulate some more facts about the geometry implied by the HJ and propagator equations. We will come back to this issue in section 4.9.

4.3 PROPERTIES OF $K_{\mu\nu}$

By differentiating the HJ equation once more, we obtain

$$K_{\mu\nu} = -G^{AB}\partial_\nu v_{A\mu}v_B \quad . \quad (4.13)$$

To extract information from this equation, we proceed as follows (similar arguments will be used repeatedly in the following). Since the v_A and $v_{A\mu}$ provide a basis for (co-)vectors on \mathcal{M} (for any x), in particular the 10 covectors $\partial_\nu v_{A\mu}$ can be expanded in this basis. Thus we have

$$\partial_\nu v_{A\mu}(x) = C_{\mu\nu}^\lambda(x)v_{A\lambda}(x) + D_{\mu\nu}(x)v_A(x) \quad (4.14)$$

for some coefficients $C_{\mu\nu}^\lambda(x)$ and $D_{\mu\nu}(x)$. Since $\partial_\nu v_{A\mu}(x) = \partial_\mu v_{A\nu}(x)$, these coefficients are symmetric.

Plugging this expansion into (4.9) and using the orthogonality (4.6) of v_A and $v_{A\mu}$, one finds

$$D_{\mu\nu}(x) = -K_{\mu\nu}(x) \quad . \quad (4.15)$$

Since $K_{\mu\nu}$ is a tensor with respect to x , it follows that $C_{\mu\nu}^\lambda$ is (transforms as) a connection, and we will denote the corresponding covariant derivative by D_μ . Hence we can write (4.14) as

$$D_\nu v_{A\mu}(x) = -K_{\mu\nu}(x)v_A(x) \quad . \quad (4.16)$$

To see what D_μ is, we act with it on (4.9). Using (4.6), one simply finds

$$D_\lambda K_{\mu\nu} = 0 \quad . \quad (4.17)$$

Hence, since $K_{\mu\nu}$ is non-degenerate and $C_{\mu\nu}^\lambda$ is symmetric, D_μ is the unique metric-compatible and torsion-free connection associated with $K_{\mu\nu}$. For the flat metric, (4.16) reduces to (3.45).

Using this, the innocuous-looking equation (4.16) has a somewhat surprising consequence. Namely, differentiating (4.16) once more and taking commutators, one finds

$$[D_\lambda, D_\nu]v_{A\mu} = -(K_{\mu\nu}v_{A\lambda} - K_{\mu\lambda}v_{A\nu}) \quad . \quad (4.18)$$

Since the $v_{A\mu}$, regarded as (a five-parameter family of) five vectors on \mathbb{R}^4 , span the tangent space at each x , it follows that the Riemann curvature tensor of $K_{\mu\nu}$ has the form

$$R(K)_{\sigma\nu\lambda\mu} = K_{\sigma\lambda}K_{\nu\mu} - K_{\sigma\mu}K_{\nu\lambda} \quad . \quad (4.19)$$

This characterizes the round metric on the four-sphere (of unit radius). Therefore $K_{\mu\nu}$, for an arbitrary metric perturbation $h_{\mu\nu}$, can only differ from $K_{\mu\nu}^0$ by a coordinate transformation!

This can also be checked explicitly. We need to show that the metric variation of $K_{\mu\nu}$ is a diffeomorphism, i.e. that there exists a vector field ξ_μ such that

$$\delta_h K_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu . \quad (4.20)$$

To analyse the metric variation of $K_{\mu\nu}$, as defined by (4.9),

$$\delta_h K_{\mu\nu} = (\delta_h G^{AB}) v_{A\mu} v_{B\nu} + G^{AB} (\delta_h v_{A\mu}) v_{B\nu} + G^{AB} v_{A\mu} \delta_h v_{B\nu} , \quad (4.21)$$

it is convenient to expand $\delta_h v_A$ in our standard basis as

$$\delta_h v_A = C v_A + C^\nu v_{A\nu} . \quad (4.22)$$

Thus e.g.

$$C = G^{AB} v_B \delta_h v_A , \quad (4.23)$$

which, using $\delta_h (G^{AB} v_A v_B) = 0$, can also be written as

$$C = -\frac{1}{2} (\delta_h G^{AB}) v_A v_B , \quad (4.24)$$

with similar expressions for C^ν .

Using

$$\delta_h v_{A\mu} = \partial_\mu \delta_h v_A \quad (4.25)$$

and (4.16), one finds that $\delta_h v_{A\mu}$ has the expansion

$$\delta_h v_{A\mu} = (\partial_\mu C - K_{\mu\nu} C^\nu) v_A + (D_\mu C^\nu + \delta_\mu^\nu C) v_{A\nu} . \quad (4.26)$$

Plugging this into (4.21) one finds after a short calculation that $\delta_h K_{\mu\nu}$ indeed takes the form (4.20), with

$$\xi_\mu = -\frac{1}{2} \partial_\mu C + K_{\mu\nu} C^\nu . \quad (4.27)$$

We thus obtain a map

$$\text{Metric Deformations on } \mathbb{R}^4 \Rightarrow \text{Coordinate Transformations on } S^4 \quad (4.28)$$

We will not pursue this matter here, but we believe that a better understanding of this map (and appreciation of the *raison d'être* of its existence) should lead to a more geometrical understanding of the behaviour of instantons under metric variations.

Since the metric $K_{\mu\nu}$ is equivalent to the round metric on the four-sphere for all values of the moduli y^A , it must also be true that

$$\partial_A K_{\mu\nu} = D_\mu \eta_{A\nu} + D_\nu \eta_{A\mu} \quad (4.29)$$

for some vector field $\eta_{A\mu}$. This equation, which we will establish below, once again displays the intricate way in which the geometry of the moduli space interacts with that of the space-time.

To determine $\partial_A K_{\mu\nu}$, we need to consider the covariant derivative $\nabla_A v_B$. As this is a (symmetric) two-tensor on \mathcal{M} , it can be expanded as a bilinear in our basis $(v_A, v_{A\mu})$. But since

$$v^A \nabla_A v_B = v^A \nabla_B v_A = 0 \quad , \quad (4.30)$$

$\nabla_A v_B$ is orthogonal to v_A and v_B and must therefore have an expansion of the form

$$\nabla_A v_B = L^{\mu\nu} v_{A\mu} v_{B\nu} \quad , \quad (4.31)$$

where $L^{\mu\nu}$ is symmetric. For the flat metric one has

$$L_0^{\mu\nu} = -K_0^{\mu\nu} \quad , \quad (4.32)$$

so that, using (4.12), we obtain the equation (2.32), namely

$$\nabla_A^0 v_B^0 = -G_{AB}^0 + v_A^0 v_B^0 \quad . \quad (4.33)$$

Using the definition of $L_{\mu\nu}$, one can now determine

$$\partial_A K_{\mu\nu} = -2L_{\mu\nu} v_A + v_A^\lambda (D_\mu L_{\lambda\nu} + D_\nu L_{\lambda\mu}) \quad (4.34)$$

(where we have raised and lowered indices using the metric $K_{\mu\nu}$). This essentially identifies $L_{\mu\nu}$ and its covariant derivatives (with respect to x) with components of the Christoffel symbols of the moduli space metric. We will make this correspondence more precise below.

Using (4.16), it now follows that $\partial_A K_{\mu\nu}$ can be written as

$$\partial_A K_{\mu\nu} = D_\mu (v_A^\lambda L_{\lambda\nu}) + D_\nu (v_A^\lambda L_{\lambda\mu}) \quad (4.35)$$

so that

$$\eta_{A\mu} = v_A^\lambda L_{\lambda\mu} \quad . \quad (4.36)$$

4.4 THE CURVATURE AND THE ROLE OF THE PROPAGATOR AND EINSTEIN EQUATIONS

So far, everything we have derived is a consequence of the HJ equation (4.3) alone. We now come to the heart of the matter, namely the interplay of the resulting geometry with the propagator and Einstein equations (4.2, 4.1).

We first note that, given the HJ equation, the propagator equation is equivalent to

$$\square F^2 = 0 \Leftrightarrow \nabla^A v_A = -4 \Leftrightarrow K_{\mu\nu} L^{\mu\nu} = -4 \quad . \quad (4.37)$$

Note that for the flat metric this is identically satisfied because $L_0^{\mu\nu} = -K_0^{\mu\nu}$ (4.32).

Also, given the HJ equation the Einstein equation is equivalent to

$$R_{AB} = -4G_{AB} \Leftrightarrow R_{AB} v^A(x) v^B(x) = -4 \quad \forall x \quad . \quad (4.38)$$

Let us therefore calculate the Ricci tensor

$$\begin{aligned} R_{AB} v^A v^B &= v^B [\nabla_A, \nabla_B] v^A \\ &= v^B \nabla_A \nabla_B v^A - v^B \nabla_B \nabla_A v^A \\ &= -(\nabla_A v_B)(\nabla^A v^B) - v^B \nabla_B \nabla_A v^A \quad . \end{aligned} \quad (4.39)$$

In passing to the last line we have used the symmetry $\nabla_A v_B = \nabla_B v_A$ and (4.30). Expressing this in terms of $L^{\mu\nu}$, we obtain the key equation

$$R_{AB} v^A v^B = -L^{\mu\nu} L_{\mu\nu} - v^B \nabla_B (L_{\mu\nu} K^{\mu\nu}) \quad (4.40)$$

relating the propagator and Einstein equations.

We see that if the second term on the right hand side is zero, e.g. if $\square F^2 = 0$, then the Ricci tensor is negative (as a quadratic form). To first order in the metric perturbation we can significantly sharpen this statement to the statement that the metric is actually Einstein.

Indeed, since to zero'th order in the metric perturbation, i.e. for the flat metric, we have (4.32), to $\mathcal{O}(h)$ in the metric perturbation we can expand $L_{\mu\nu}$ as

$$L_{\mu\nu} = -K_{\mu\nu} + \ell_{\mu\nu} \quad (4.41)$$

where $\ell_{\mu\nu}$ is of $\mathcal{O}(h)$ (we could alternatively have expanded $L_{\mu\nu} = -K_{\mu\nu}^0 + \tilde{\ell}_{\mu\nu}$ with $\tilde{\ell}_{\mu\nu}$ also of $\mathcal{O}(h)$).

Since we know that the propagator equation is true to $\mathcal{O}(h)$, we learn that to this order the second term in (4.40) is absent. We also learn that $\ell_{\mu\nu}$ is traceless (with respect to either $K_{\mu\nu}$ or $K_{\mu\nu}^0$ - to this order this is the same),

$$\mathcal{O}(h) : \quad K^{\mu\nu} L_{\mu\nu} = -4 \Leftrightarrow K^{\mu\nu} \ell_{\mu\nu} = 0 \quad . \quad (4.42)$$

But this now implies that

$$L^{\mu\nu} L_{\mu\nu} = 4 + \mathcal{O}(h^2) \quad , \quad (4.43)$$

so that

$$R_{AB}v^Av^B = -4 + \mathcal{O}(h^2) . \quad (4.44)$$

Thus we see that to $\mathcal{O}(h)$ the HJ and propagator equations imply $L^{\mu\nu}L_{\mu\nu} = 4$ and hence the Einstein equation! Conversely, however, the HJ equation and the Einstein equation do not imply the propagator equation.

Now let us assume, for argument's sake, that all three equations actually continue to hold to $\mathcal{O}(h^2)$. In this case we can still expand $L_{\mu\nu}$ as before, and the propagator equation is satisfied if $\ell_{\mu\nu}$ is traceless with respect to $K_{\mu\nu}$. But now for the Einstein equation to hold to $\mathcal{O}(h^2)$, we need

$$L^2 = 4 + \mathcal{O}(h^3) . \quad (4.45)$$

Thus the $\mathcal{O}(h)$ -piece $\ell_{\mu\nu}^1$ of $\ell_{\mu\nu}$ has to be zero (otherwise there would be a non-zero contribution to L^2 of $\mathcal{O}(h^2)$), and we obtain the result that

$$L_{\mu\nu} = -K_{\mu\nu} + \mathcal{O}(h^2) . \quad (4.46)$$

We will now show that this leads to a contradiction at $\mathcal{O}(h)$, namely that this would imply that the Information Metric is maximally symmetric for an arbitrary boundary metric perturbation.

Indeed, the same argument that leads to (4.33) shows that if $L_{\mu\nu} = -K_{\mu\nu}$ to some order, then

$$\nabla_A v_B = -G_{AB} + v_A v_B \quad (4.47)$$

to that order. But then the argument leading to (2.37) implies that the bulk metric is maximally symmetric for an arbitrary $\mathcal{O}(h)$ variation of the boundary metric. Obviously for a number of reasons this cannot be true - it is enough to think of symmetries and Killing vectors. Alternatively, it follows from the Fefferman-Graham analysis that a maximally symmetric bulk metric will induce a conformally flat metric on the boundary - see [20] for an explicit proof.

Thus we learn that not all three of our basic equations (4.1, 4.2, 4.3) can be true to $\mathcal{O}(h^2)$.

4.5 A FOUR-PARAMETER FAMILY OF COORDINATE SYSTEMS ON \mathcal{M}

We have seen that it is quite useful to adopt the frame (actually, four-parameter family of frames) $(v_A, v_{A\mu})$ as a basis of (co-)tangent vectors on \mathcal{M} . This can more succinctly be understood as a change of coordinates

$$y^A = (\rho, a^\mu) \rightarrow z^M(y^A) \quad (4.48)$$

on \mathcal{M} . Indeed, since $v_A = \frac{1}{4}\partial_A \log F^2$ and $v_{A\mu} = \partial_A \partial_\mu \log F^2$, the change of coordinates in question is

$$(\rho, a^\mu) \rightarrow (\tfrac{1}{4} \log F^2(x; \rho, a), \tfrac{1}{4} \partial_\mu \log F^2(x; \rho, a)) . \quad (4.49)$$

Raising the index of $\partial_\mu \log F^2$ with the boundary space-time metric $g_{\mu\nu}$, the four-parameter family of coordinate transformation that we will actually consider is

$$y^A = (\rho, a^\mu) \rightarrow z^M(x) \equiv (r(x), z^\mu(x)) = (\tfrac{1}{4} \log F^2(x; \rho, a), \tfrac{1}{4} g^{\mu\nu} \partial_\nu \log F^2(x; \rho, a)) . \quad (4.50)$$

It follows from (4.12) that in these coordinates the Information Metric takes the simple form

$$ds^2 = G_{AB} dy^A dy^B = dr^2 + \gamma_{\mu\nu} dz^\mu dz^\nu , \quad (4.51)$$

where

$$\gamma_{\mu\nu} = g_{\mu\rho} g_{\nu\lambda} K^{\rho\lambda} . \quad (4.52)$$

The HJ equation $G_{AB} v^A v^B = 1$ is now simply the statement that $G_{r(x)r(x)} = 1$ for all x and (4.6) says that the off-diagonal component $G_{r(x)z^\mu(x)} = 0$.

Note that if the propagator equation $\square F^2 = 0$ holds in addition to the HJ equation, then the functions $\exp 4r$ and z^μ are harmonic so that these are harmonic coordinates for the Information Metric! This will be useful later.

Note also that for the flat metric (and the choice $x = 0$) the coordinate transformation

$$(\rho, a^\mu) \rightarrow (e^{r_0} = (F_0^2)^{1/4}, \tfrac{1}{8} \partial_\mu \log F_0^2) \quad (4.53)$$

is just the inversion isometry

$$(\rho, a^\mu) \rightarrow (\tfrac{\rho}{a^2 + \rho^2}, \tfrac{a^\mu}{a^2 + \rho^2}) \quad (4.54)$$

of the standard AdS metric, now more invariantly expressed in terms of F_0^2 . This is reflected in the fact that in the z^M -coordinates (for any value of x) the unperturbed Information Metric takes the standard AdS form

$$ds_0^2 = dr_0^2 + \tfrac{1}{4} e^{-2r_0} \delta_{\mu\nu} dz^\mu dz^\nu . \quad (4.55)$$

4.6 THE GEODESIC DISTANCE ON \mathcal{M} AND THE INSTANTON DENSITY

Writing the metric in the form (4.55) or (in general) (4.51) makes it manifest that $r = \frac{1}{4} \log F^2(x; \rho, a)$ is the geodesic distance along paths of constant z^μ . It is also the regularized geodesic distance between points with different values of z provided that for

one of the end-points $r \rightarrow \infty$ because the difference between the z 's becomes irrelevant in this limit - in some sense $r = \infty$ is a single point.

An alternative way of establishing this result is to use essentially *verbatim* the argument outlined in (2.54)-(2.59) which only relies on the validity of the equation $G^{AB}v_A v_B = 1$ which thus once again can be interpreted as the HJ equation.

We have thus shown that to the order to which the HJ equation holds the instanton density F^2 is directly related to the classical geodesic distance of a boundary-to-bulk path in the geometry provided by the Information Metric. One would dearly like to have a more conceptual explanation for this.

4.7 GAUSS-CODAZZI EQUATIONS FOR THE INFORMATION METRIC ON \mathcal{M}

Using the coordinates (r, z^μ) , equation (4.34) now has a much more transparent interpretation. In particular, since

$$\partial_r = v^A \partial_A \quad , \quad (4.56)$$

we have

$$\partial_r K_{\mu\nu} = -2L_{\mu\nu} \quad . \quad (4.57)$$

Thus $L_{\mu\nu}$ is essentially the extrinsic curvature (second fundamental form) of $\gamma_{\mu\nu}$, defined by

$$\Theta_{\mu\nu} = \frac{1}{2} \partial_r \gamma_{\mu\nu} \quad , \quad (4.58)$$

the precise relation being

$$\Theta_{\mu\nu} = g_{\mu\rho} g_{\nu\lambda} L^{\rho\lambda} \quad . \quad (4.59)$$

For the unperturbed metric in the form (4.55) one evidently has

$$\Theta_{\mu\nu}^0 = -\gamma_{\mu\nu}^0 \quad , \quad (4.60)$$

which is simply the equation $L_{\mu\nu}^0 = -K_{\mu\nu}^0$ (4.32).

The other component of (4.34), obtained by contracting $\partial_A K_{\mu\nu}$ with v_λ^A instead of v^A , then essentially says that the Christoffel symbols of $\gamma_{\mu\nu}$, i.e. z^λ -derivatives of $\gamma_{\mu\nu}$, can be expressed as covariant x -derivatives of the extrinsic curvature $\Theta_{\mu\nu}$!

The Gauss-Codazzi equations express the curvature of the Information Metric G_{MN} in terms of the curvature of $\gamma_{\mu\nu}$ and the extrinsic curvature $\Theta_{\mu\nu}$ and its trace $\Theta = \gamma^{\mu\nu} \Theta_{\mu\nu}$. The expressions for the Ricci tensor are

$$R(G)_{rr} = -\partial_r \Theta - \Theta^{\mu\nu} \Theta_{\mu\nu} \quad (4.61)$$

$$R(G)_{r\nu} = \nabla^\mu \Theta_{\mu\nu} - \partial_\nu \Theta \quad (4.62)$$

$$R(G)_{\mu\nu} = R(\gamma)_{\mu\nu} - \Theta \Theta_{\mu\nu} + 2\Theta_{\mu\lambda} \Theta_\nu^\lambda - \partial_r \Theta_{\mu\nu} \quad . \quad (4.63)$$

It follows that the scalar curvature of G_{MN} is

$$R(G) = R(\gamma) - \Theta^2 - \Theta_{\mu\nu}\Theta^{\mu\nu} - 2\partial_r\Theta \quad . \quad (4.64)$$

In particular, we now recognise the key equation (4.40) as nothing other than the (rr) -component (4.61) of the Gauss-Codazzi equation. Let us now analyse these equations to first order in the metric perturbation. Since $\Theta_{\mu\nu}^0 = -\gamma_{\mu\nu}^0$, we can expand $\Theta_{\mu\nu}$ like $L_{\mu\nu}$ as

$$\Theta_{\mu\nu} = -\gamma_{\mu\nu} + \theta_{\mu\nu} \quad . \quad (4.65)$$

Given the HJ equation, the propagator equation $\square F^2 = 0$ is equivalent to $K_{\mu\nu}L^{\mu\nu} = -4$. In the present language this is the statement

$$\square F^2 = 0 \Leftrightarrow \Theta = -4 \quad . \quad (4.66)$$

Thus we see that to first order in the metric perturbation, $\theta_{\mu\nu}$ (like $\ell_{\mu\nu}$) is traceless. Therefore to this order we also have

$$\Theta^{\mu\nu}\Theta_{\mu\nu} = 4 + \mathcal{O}(h^2) \quad , \quad (4.67)$$

and we find that the (rr) -component of the Einstein equation

$$R_{rr} = -4G_{rr} = -4 \quad (4.68)$$

is satisfied. Since it holds for all x we know that all the other components of the Einstein equation are also satisfied - this is just what we found before: to $\mathcal{O}(h)$, the HJ and propagator equations imply the Einstein equation.

However, we also learn one new fact about the curvature of $\gamma_{\mu\nu}$. Namely, using (4.64) and $R_{AB} = -4G_{AB}$, so that $R(G) = -20$, we see that

$$R(\gamma) = 0 \quad . \quad (4.69)$$

Thus to first order $\gamma_{\mu\nu}$ is a scalar-flat perturbation of the flat (with respect to the z^μ) metric $\gamma_{\mu\nu}^0$. Note that, even though it is true that (4.51) is Einstein if

$$\gamma_{\mu\nu}(r, z) = e^{\pm 2r} \tilde{\gamma}_{\mu\nu}(z) \quad (4.70)$$

with $\tilde{\gamma}_{\mu\nu}(z)$ Ricci flat (hence the Fefferman-Graham expansion is trivial in this case), there is no reason to expect the metric $\gamma_{\mu\nu}$ to be Ricci-flat in general - there are simply not enough Ricci-flat perturbations of the flat metric in Euclidean space. In fact there are none which are asymptotically flat while every transverse traceless perturbation of the flat metric is automatically scalar flat - see (4.81) below.

Some further information about $\gamma_{\mu\nu}$ and its curvature $R(\gamma)_{\mu\nu}$ can be obtained by looking at the other components of the Gauss-Codazzi equations, but we will not pursue this here.

4.8 THE RELATION BETWEEN \sqrt{K} AND THE INSTANTON DENSITY

In this section we will establish an identity which is needed to show directly the equivalence between the integral (2.11) and algebraic (4.12) forms of the Information Metric. The required identity is

$$\sqrt{K} = 16\sqrt{g}F^2 . \quad (4.71)$$

To zero'th order in the metric perturbation we indeed have the relationship

$$\sqrt{K_0} = 16F_0^2 \quad (4.72)$$

(see (4.8)). We will now establish that this is yet one more equation which continues to hold to first order in the metric perturbation, i.e. that (4.71) is valid to $\mathcal{O}(h)$.

First of all we note that the integrated version of (4.71),

$$\int \sqrt{K} = 16 \int \sqrt{g}F^2 , \quad (4.73)$$

is automatically true. This follows because $K_{\mu\nu}$ is equivalent to the round metric on the four-sphere for all metric perturbations - see (4.20). Thus we have

$$\int \sqrt{K} = \int \sqrt{K_0} = \frac{8\pi^2}{3} . \quad (4.74)$$

Moreover, and rather more obviously,

$$\int \sqrt{g}F^2 = \int F_0^2 = \frac{\pi^2}{6} , \quad (4.75)$$

as the instanton number cannot change, i.e. $\delta_h F^2$ is a total derivative.

To establish (4.71), we proceed in three steps. We first show that the r -dependence (i.e. F^2 -dependence) of \sqrt{K} is as given in (4.71). This is easy. We then show that \sqrt{K} is independent of the z^μ . This requires a bit more work. Lastly we show that the normalization of $\int \sqrt{K}$ then fixes the possibly x -dependent factor between the left- and right-hand sides of (4.71) to be equal to the constant 16. This is once again easy.

To determine the r -dependence of \sqrt{K} , we use (4.57) which implies

$$K^{\mu\nu} \partial_r K_{\mu\nu} = -2K^{\mu\nu} L_{\mu\nu} . \quad (4.76)$$

Since the HJ equation holds to $\mathcal{O}(h)$, we have $K^{\mu\nu} L_{\mu\nu} = -4$ and thus

$$\partial_r \sqrt{K} = 4\sqrt{K} , \quad (4.77)$$

which implies

$$\sqrt{K} = \sqrt{g}F^2 f(z, x) , \quad (4.78)$$

where $f(z, x)$ is some function of the coordinates z^μ and x^μ . We have introduced \sqrt{g} on the right-hand-side because \sqrt{K} is a density with respect to x^μ and any such scalar density can be written as $\sqrt{g(x)}$ times a function. Alternatively, note that if we take our metric perturbation to be traceless (by conformal invariance), then $\sqrt{g} = 1$.

To determine the z -dependence, we make use of two facts: that the z^μ are harmonic coordinates and that the scalar curvature $R(\gamma_{\mu\nu}) = 0$. We had already noted before that the propagator equation implies that the z^μ are harmonic with respect to the bulk \square . Due to the special form (4.51) of the metric in (r, z) -coordinates, this implies that the z^μ are also harmonic with respect to the Laplacian \square^γ associated to $\gamma_{\mu\nu}$. Indeed, denoting the Christoffel symbols of G_{MN} and $\gamma_{\mu\nu}$ by Γ_{MNP} and $\gamma_{\mu\nu\lambda}$ respectively, we have

$$\square z^\mu = 0 \Leftrightarrow G^{NP} \Gamma_{NP}^\mu = 0 \Leftrightarrow \gamma^{\nu\lambda} \gamma_{\nu\lambda}^\mu = 0 \Leftrightarrow \square^\gamma z^\mu = 0 \quad . \quad (4.79)$$

Since the unperturbed metric $\gamma_{\mu\nu}^0$ is the flat metric, $\gamma_{\mu\nu\lambda}^0 = 0$ and the harmonic gauge condition becomes

$$\partial_{z^\nu} \gamma_1^\nu{}_\mu = \frac{1}{2} \partial_{z^\mu} \gamma_1 \quad , \quad (4.80)$$

where indices have been raised and lowered with $\gamma_{\mu\nu}^0$ and γ_1 is the trace of the first-order perturbation $\gamma_{\mu\nu}^1$ of $\gamma_{\mu\nu}^0$.

In the harmonic gauge, the general expression for the linearized Ricci scalar

$$R(\gamma_{\mu\nu}) = \partial_{z^\mu} \partial_{z^\nu} \gamma_1^{\mu\nu} - \gamma_0^{\mu\nu} \partial_{z^\mu} \partial_{z^\nu} \gamma_1 \quad (4.81)$$

becomes

$$R(\gamma_{\mu\nu}) = -\frac{1}{2} \gamma_0^{\mu\nu} \partial_{z^\mu} \partial_{z^\nu} \gamma_1 \quad . \quad (4.82)$$

Thus vanishing of the Ricci scalar implies

$$R(\gamma_{\mu\nu}) = 0 \Rightarrow \partial_{z^\mu} \gamma_1 = 0 \quad , \quad (4.83)$$

because for a small metric perturbation $\partial_{z^\mu} \gamma_1$ necessarily goes to zero at infinity.

To relate this to a statement about $K_{\mu\nu}$ we use (4.52) to deduce

$$\partial_{z^\mu} \gamma_1 = 0 \Rightarrow \partial_{z^\mu} K_1 = 0 \quad , \quad (4.84)$$

where K_1 is the trace of the first order perturbation (4.20),

$$K_{\mu\nu}^1 = D_\mu^0 \xi_\nu + D_\nu^0 \xi_\mu \quad , \quad (4.85)$$

of $K_{\mu\nu}$. Hence

$$\partial_{z^\mu} \sqrt{K} = 0 \quad (4.86)$$

and

$$\sqrt{K} = \sqrt{g}F^2 f(x) . \quad (4.87)$$

Now to zero'th order we have $f(x) = 16$. We thus write

$$f(x) = 16 + \phi(x) \quad (4.88)$$

where $\phi(x)$ is of $\mathcal{O}(h)$. It now follows from (4.73) that

$$\int F_0^2 \phi(x) = 0 . \quad (4.89)$$

But since F_0^2 is the AdS boundary-to-bulk scalar propagator, this implies that $\phi(x) = 0$. Hence we have established (4.71).

Incidentally note that this calculation can also be read as establishing a relationship between the vector field ξ^μ appearing in the variation of $K_{\mu\nu}$ under metric variations, and the vector field Y^μ defined by

$$\delta_h F^2 = \partial_\mu Y^\mu . \quad (4.90)$$

4.9 THE EQUIVALENCE OF THE INTEGRAL AND ALGEBRAIC FORMS OF THE INFORMATION METRIC

We have encountered two different forms of the Information Metric, the original integral expression (2.11) which, in order to avoid confusion we will in this subsection denote by G_{AB}^{int} ,

$$G_{AB}^{\text{int}} = \frac{15}{8\pi^2} \int \sqrt{g} F^2 \partial_A \log F^2 \partial_B \log F^2 = \frac{30}{\pi^2} \int \sqrt{g} F^2 v_A v_B , \quad (4.91)$$

and the algebraic (vielbein) representation of the metric (4.12), i.e.

$$G_{AB} = v_A v_B + K^{\mu\nu} v_{A\mu} v_{B\nu} , \quad (4.92)$$

which could e.g. have been obtained by using (3.54). We now want to show directly that these two are equivalent. As G_{AB} is x -independent, we can write

$$\begin{aligned} G_{AB} &= \frac{6}{\pi^2} \int \sqrt{g} F^2 G_{AB} \\ &= \frac{6}{\pi^2} \int \sqrt{g} F^2 (v_A v_B + K^{\mu\nu} v_{A\mu} v_{B\nu}) \\ &= \frac{1}{5} G_{AB}^{\text{int}} + \frac{6}{\pi^2} \int \sqrt{g} F^2 K^{\mu\nu} v_{A\mu} v_{B\nu} . \end{aligned} \quad (4.93)$$

We thus need to show that the second term equals $(4/5)G_{AB}^{\text{int}}$. To proceed, we use (4.71) to rewrite this as

$$\int \sqrt{g} F^2 K^{\mu\nu} v_{A\mu} v_{B\nu} = \int \sqrt{K} K^{\mu\nu} v_{A\mu} v_{B\nu} \quad (4.94)$$

We now integrate by parts to obtain

$$\int \sqrt{g} F^2 K^{\mu\nu} v_{A\mu} v_{B\nu} = - \int \sqrt{g} F^2 K^{\mu\nu} v_A D_\mu v_{B\nu} . \quad (4.95)$$

Using (4.16), we learn that

$$- \int \sqrt{g} F^2 K^{\mu\nu} v_A D_\mu v_{B\nu} = 4 \int \sqrt{g} F^2 v_A v_B . \quad (4.96)$$

Therefore this term gives precisely the missing contribution to G_{AB}^{int} and we have shown that

$$G_{AB} = G_{AB}^{\text{int}} . \quad (4.97)$$

To reiterate: this means that we do not have to define the variation of the Information Metric through the variation of the x -integral (2.11). Instead we can extract the metric variation from the variation (3.54) of the HJ equation, use that expression to verify the propagator (and thus the Einstein) equation, and we can prove (as we just did) that the metric variation obtained in this (much simpler) way is indeed the variation of (2.11).

5 CONCLUSIONS

In this paper we have shown that, to first order in the perturbation of the boundary space-time metric, the Information Metric on the 5-dimensional moduli space of $k = 1$ $SU(2)$ instantons is Einstein. Furthermore, to this order the perturbed instanton action density is the corresponding boundary-to-bulk massless scalar propagator and, quite remarkably, the regularized boundary-to-bulk geodesic distance is proportional to the logarithm of the perturbed instanton action density.

These results show that it is rather compelling to think of the bulk space-time of the AdS_5/CFT_4 correspondence as *being* the instanton moduli space. Indeed, at least to this order, physically relevant quantities of the non-trivial perturbed bulk space-time like massless and massive scalar propagators and the geodesic distance are then directly related to the simplest and most natural function on the bulk space-time, namely the instanton density itself.

It would be nice to know if there is a similarly fruitful or suggestive reinterpretation of the AdS/CFT bulk space-time in other dimensions. For example, as for the unperturbed instanton moduli space, symmetry arguments imply that the Information Metric on the moduli space of degree one rational maps from the two-sphere to itself is the AdS_3 -metric [12]. In [12] it is also shown that the Information Metric on the $(4k - 1)$ -dimensional space of rational maps of degree k is non-degenerate for $k > 1$. One can also ask

if perhaps AdS_7 emerges as a moduli space of a 6-dimensional theory of interacting anti-self-dual tensor fields.

The results we have obtained, although to a certain (limited) extent anticipated from physics considerations, are certainly quite surprising from a mathematical point of view. One would like to gain a better mathematical insight in order to obtain more elegant proofs of these statements. Note that the (rather cumbersome) proofs given in this paper rely on the explicit form of the scalar instanton Green's functions. One would like to be able to prove these statements in a more abstract way, ideally by just making use of the self-duality equations. It would also be good to have a deeper understanding of the significance of the map from metric deformations on \mathbb{R}^4 to diffeomorphisms of S^4 we obtained in section 4.3. Any such conceptual progress should help in analyzing the higher order corrections and to see which of the above statements, if any, continue to hold to higher orders in the metric perturbation.

Since the only information we have at the moment is that to first order the propagator and Hamilton-Jacobi equation imply that the Information Metric is Einstein and that to second order not all the three statements can be true simultaneously, it would also be useful to be able to analyze the Information metric for a tractable (i.e. sufficiently symmetric) metric on S^4 which is far from the (conformal class of the) standard metric.

As we discussed in the introduction, the physics intuition which led us to analyze this problem was based on the AdS/CFT correspondence together with the D-instanton probe idea. However, our results are neither a test nor a consequence of the AdS/CFT correspondence but rather logically independent of it. In particular, there are two important differences between this work and the usual AdS/CFT scenario.

Firstly, the AdS/CFT correspondence is supposed to hold for $\mathcal{N} = 4$ $SU(N)$ gauge theories in the large N limit whereas here we are dealing just with the $SU(2)$ theory.¹ Secondly, the Information Metric is certainly not the metric on the gauge fields that one normally uses in studying Yang-Mills theories. The usual metric is the L^2 -metric which, as pointed out earlier, is not conformally invariant.

Perhaps the resolution to both the seeming differences lies in taking the large N limit where one integrates out all the zero modes of the instanton associated to the gauge orientations (whose number grows linearly with N), leaving behind just the $SU(2)$ instanton moduli which in turn are identified with the bulk space-time [6].

¹Actually we were only considering pure $SU(2)$ Yang-Mills because we were interested in the deformation of the AdS_5 part of the bulk which only sees the bosonic moduli of the super-instanton. However throughout this paper we have been assuming an underlying $\mathcal{N} = 4$ structure justifying the semi-classical approximation around instanton solutions.

One way to test this would be to start with a 5-dimensional $SU(N)$ theory reduced to one dimension on an instanton solution, with the moduli being slowly varying functions of this coordinate. This reduction of course gives the L^2 -metric for the kinetic terms of the moduli. Integrating out the moduli associated with gauge orientation should now give rise to an effective kinetic term for the remaining moduli (namely those of the $SU(2)$ instantons). The task would then be to check if in the large N limit this reduces to the Information Metric once all the other fields have been integrated out, as might be expected on the basis of 4-dimensional conformal invariance.

If this is true, then one might start from an $\mathcal{N} = (4, 4)$ sigma model on the large N $SU(N)$ instanton moduli space obtained by reducing a 6-dimensional theory on a single instanton. The large N limit of this sigma model should then give a definition of a string moving in $AdS_5 \times S^5$. This is somewhat reminiscent of the Matrix theory description of the $(2, 0)$ -theory [21] and related suggestions in the AdS/CFT context [22].

In any case, the results of this paper seem to suggest that the one-instanton sector of $SU(2)$ gauge theory on a space which is topologically S^4 or \mathbb{R}^4 gives rise to a theory of gravity on the instanton moduli space similar to the way that string theory produces gravity on the target space. In particular, in string theory the target space is also precisely the moduli (zero-mode) space of the 2-dimensional world-sheet theory. Moreover, the criterion of conformal invariance, which leads to the target space equations of motion in string theory via the β -function equations, here led us to choose the Information Metric on the moduli space which then turned out to satisfy the (linearized) Einstein equations. In this sense, our construction has some similarities with attempts to use Holographic Renormalization Group ideas to (re-)construct the bulk space-time from the boundary field-theory data - see e.g. [23, 24].

The Information Metric (or more precisely its first order deformation $\delta_h G_{AB}$) and $\int d^4x \phi(x) \text{tr} F^2(x)$ are certain operators in the gauge theory whose semiclassical expectation values define the on-shell linearized graviton and massless scalar field on the instanton moduli space. Thinking along these lines, there are a number of questions one might try to answer.

There are certainly infinitely many operators that one can construct on the gauge theory side that are gauge and conformally invariant. What other (perhaps massive) fields do they define in the bulk theory? On the gauge theory side we can compute correlation functions of the above operators beyond the semiclassical approximation. What would this correspond to in the bulk theory? These quantities would be non-local on the boundary but local in the instanton moduli space. Does this therefore produce interaction vertices in the bulk theory? What is the role of multi-instantons - multi-particle states? What computation in the gauge theory would yield information on quantum

gravity effects in the bulk? (We most certainly do not expect these to arise from a summation over world-volume topologies as in string theory!) Etc. We believe that it is worthwhile to attempt to understand these issues.

ACKNOWLEDGEMENTS

We would like to thank Edi Gava and Martin O'Loughlin for many useful discussions. This research is partially supported by EC contract no. CT-2000-00148.

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